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SPHERICALLY SYMMETRIC WAVES OF A REACTION-DIFFUSION EQUATION.(U)
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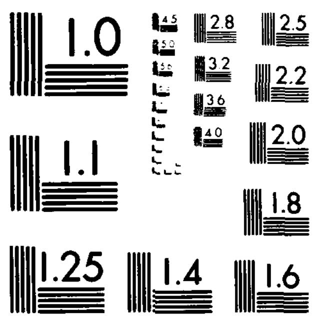
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SPHERICALLY SYMMETRIC WAVES OF A REACTION-DIFFUSION EQUATION

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ABSTRACT

Attention is focused on the scalar reaction-diffusion equation: $u_t = \Delta_x u + f(u)$, where f is cubic-like and $f(0) = f(1) = 0$. Amongst spherically symmetric solutions it is proved there is a bounded unstable equilibrium which is decreasing in radial profile. Under a concavity assumption on f this equilibrium is unique. Moreover there is a unique expanding spherical wave which is defined for all time, positive and negative. As $t \to -\infty$ this wave approaches the unstable equilibrium.

Aronson and Weinberger [2] have proved before that, in all space dimensions, there are non-trivial solutions that propagate ($u(x,t) \to 1$ uniformly on compact sets as $t \to +\infty$) and ones that decay ($u(x,t) \to 0$ uniformly as $t \to +\infty$). This suggested the existence of the unstable equilibrium.

There is an interesting global description of this propagation/decay effect. The set of initial data whose associated solutions approach the unstable equilibrium as $t \to +\infty$ splits a natural set of functions into two sets. Data from one set yields a solution that propagates, and data from the other set, a solution that decays. This fact is closely related to the uniqueness of the expanding spherical wave.

AMS (MOS) Subject Classifications: 34B25, 34C35, 35B40, 35K55.

Key Words: Reaction-diffusion equation, spherically symmetric wave, travelling wave, equilibrium solution, compact-open topology.

Work Unit Number 1: Applied Analysis

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SIGNIFICANCE AND EXPLANATION

Reaction-Diffusion Equations have been used to model nerve impulse propagation and spatially inhomogeneous situations in chemically reacting systems and population genetics. The kind of solutions that are often of interest in these areas are wave-like and do not die out.

If the underlying spatial domain for the equation is one-dimensional, the description of such wave behaviour amounts to finding a travelling wave. This is a solution whose evolution in time under the equation is given by translating along the axis. A nerve impulse is an example of a travelling wave.

If a wave in higher dimensions is expected, for instance one that is a function only of radius in a spherical co-ordinate system, this mathematical approach is not available. Such spherical waves look like one dimensional waves a long way out. In this report we invert this idea and deduce spherical wave behaviour in a model example where there is a stable one dimensional travelling wave. This approach also determines the remaining work that is necessary for a complete picture of the spherical wave behaviour for this equation.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

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SPHERICALLY SYMMETRIC WAVES OF A REACTION-DIFFUSION EQUATION

Christopher K. R. T. Jones

Chapter 1

Introduction

I. MOTIVATION

Reaction - Diffusion equations often exhibit persistent wave-like behaviour. Examples of this are the travelling pulses of the Fitzhugh-Nagumo and Hodgkin-Huxley equations, travelling fronts in some scalar equations such as the Fisher equation and target patterns and spiral waves in some models of the Belousov-Zhabotinskii reaction. Such phenomena as these usually depend on the underlying spatial domain being unbounded. This is in sharp contrast to the case of scalar reaction - diffusion equations on a bounded domain where energy arguments show that all solutions decay to some equilibrium.

The wave phenomena in one space dimension, such as the travelling pulses and fronts mentioned above, are often mathematically tractable. This approach is to find an appropriate solution to the travelling wave equations, which are ordinary differential equations, and then try to prove that they are stable as solutions to the full partial differential equation. When genuinely higher dimensional effects are involved, as in the case of target patterns or spiral waves, this mathematical attach is not available. Results have been obtained in some model situations derived by making simplifying assumptions, as in the work of Greenberg, Hassard and Hastings [1] and Kopell and Howard [1,2,3]. The main difficulty appears to be that, in general, there is not a specific mathematical object, like a travelling wave, whose discovery would prove the existence and supply a description of such behaviour.

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In this work we develop a technique for handling spherically symmetric waves of reaction - diffusion equations. If a system has spherically symmetric waves then it would also have some corresponding one-dimensional waves, loosely speaking, these would be the spherically symmetric waves "at infinity". The technique is to invert this idea and deduce the existence of spherical type wave behaviour from knowing one-dimensional behaviour. We hope that this approach will give some general conditions under which such higher dimensional wave behaviour exists but it cannot give details about the formation of these waves; this is an inherently higher-dimensional problem and could not be deduced from knowledge of the one-dimensional mechanisms.

We focus our attention on the bistable equation which is described in section III below, since that equation has the useful property of possessing an unambigously stable travelling wave. The result of applying the technique to this equation is that it guarantees the existence of spherical waves that are asymptotic to the one-dimensional travelling wave at infinity and also isolates the work that needs to be done to get a complete picture of the phenomenon; we also carry out this work.

II. GENERALITIES

In the most general case, we shall consider a system of reaction - diffusion equations of the form

$$u_t = \Delta u + f(u) \tag{1.1}$$

where $u \in \mathbb{R}^m$, $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ and $\Delta u = (\Delta u_1,\ldots,\Delta u_m)$. We shall always assume that $f : \mathbb{R}^m \to \mathbb{R}^m$ is smooth.

The natural problem for such a system on all of $\mathbb{R}^n$ is an initial-value problem, i.e. a solution $u(x,t)$ should be determined by its initial condition $u(x)$

$$u(x,0) = u(x) \quad . \tag{1.2}$$

We can therefore hope that such a system determines a semiflow on some appropriate function space. A semiflow on a space Y is a function $S : Y \times \mathbb{R}^+ \to Y$ (whose domain may not be all of $Y \times \mathbb{R}^+$, but must be an open subset) which satisfies (1) $S(S(y,t),s) = S(y,t+s)$ and (2) S is continuous on its domain. S is said to be a local semiflow if for each $y \in Y$, a set of the form $(y,[0,s))$ for some $s > 0$ is in the domain of S. S is a global semiflow if it is defined on all of $Y \times \mathbb{R}^+$.

For the equation (1.1) Y will be some function space and $S(y,t) = u(x,t)$ where $u(x,t)$ is the solution with initial condition $u(x,0) = y$. The space Y will usually be the space of bounded uniformly continuous functions $u : \mathbb{R}^n \to \mathbb{R}^m$ which will be denoted B. We are interested in locating parts of B that exhibit certain asymptotic behaviour (in particular, wave-like behaviour) when the semiflow is applied. The following definitions, due to Fife [1,2], clarify this quest.

The group of transformations that leave the Laplacian invariant, and so also the equation, are exactly the rigid motions; call this group $R(n)$. $R(n)$ acts in a natural way on B; if $T \in R(n)$ then $Tu(x) = u(Tx)$. Therefore one can see that the space $B/R(n)$ inherits the semiflow which shall also be denoted by S.

Definition 1.1. $u,v \in B/R(n)$ are said to be asymptotically equivalent if there exists a $\tau \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \|S(u,t) - S(v,t+\tau)\| = 0$$

where the norm used is the sup norm over all of $\mathbb{R}^n$.

Definition 1.2. An asymptotic state $G \subset B/R(n)$ is an equivalence class of initial data under this relation. G is said to be a stable asymptotic state (SAS) if it is open in $B/R(n)$.

As mentioned earlier one attack on reaction - diffusion equations is to look for special solutions such as equilibrium solutions, travelling waves etc. In general the least one might expect of solutions that warrant such attention is that they be defined for all $t \in \mathbb{R}$. Such a solution would certainly be "special" as the equations are parabolic and so only forward existence would be expected in general.

Definition 1.3. A solution $u(x,t)$ to (1.1) is said to be a permanent solution if it is defined for all $t \in \mathbb{R}$.

Satisfaction of the ultimate dream for a given reaction - diffusion equation would be a description of all its SAS's and the location of all their bounded permanent solutions.

We cannot usually expect to be able to prove that a certain set is a stable asymptotic state with respect to all of B, but in a given problem there may be a natural subspace of B in which the asymptotic state can easily be proved to be open. This will only be meaningful if that subspace is invariant, or at least positively invariant.

Definition 1.4. $A \subset Y$ is invariant under S, if $S(t)A = A$ for all $t \geq 0$. A is positively invariant if $S(t)A \subset A$ for all $t \geq 0$.

Suppose $A \subset B$ is positively invariant.

Definition 1.5. A set G is an SAS with respect to A if it is an asymptotic state and it is open in A.

In regard to looking for permanent solutions one of the useful ideas of dynamical systems is that of an ω-limit set. If $A \subset Y$ then

$$\omega(A) = \bigcap_{t \geq 0} cl(A \cdot [t,\infty))^{*} .$$

The crucial property of ω-limit sets is that they are closed and invariant.

A consequence of this is that points in $\omega(A)$ have local backward existence but the solutions may blow up in finite backward time, so $\omega(A)$ need not consist entirely of permanent solutions. If however we knew that S was a compact semiflow on Y (see chapter 4, section I) then $\omega(A)$ is a compact, non-empty set that consists entirely of permanent solutions.

This compactness property is not satisfied on B if it carries the sup-norm topology, however we can replace this with the compact-open topology, without destroying the semiflow property, and the

---

*for notation see section IV

compactness condition is satisfied. Permanent solutions can then be found by calculating the ω-limit set of a bounded connected [illegible] the SAS in question.

III. THE BISTABLE EQUATION

Although the main abstract construction of this work applies to any system of reaction - diffusion equations the only application worked out is to the scalar equation, called the bistable equation:

$$u_t = \Delta u + f(u) \quad , \tag{1.3}$$

where $u \in \mathbb{R}$. We split the assumptions on the nonlinearity f into two parts:

(H1) $f : \mathbb{R} \to \mathbb{R}$ is smooth. $f(0) = f(1) = 0$ and there is a unique α, between 0 and 1 such that $f(\alpha) = 0$. Furthermore $f'(0) < 0$, $f'(1) < 0$ and $\int_0^1 f(u)du > 0$.

(H2) For all $\alpha \leq \beta \leq 1$ $f''(\beta) \leq 0$.

Roughly speaking, if f satisfies (H1) then it is of the form shown in Figure 1.1

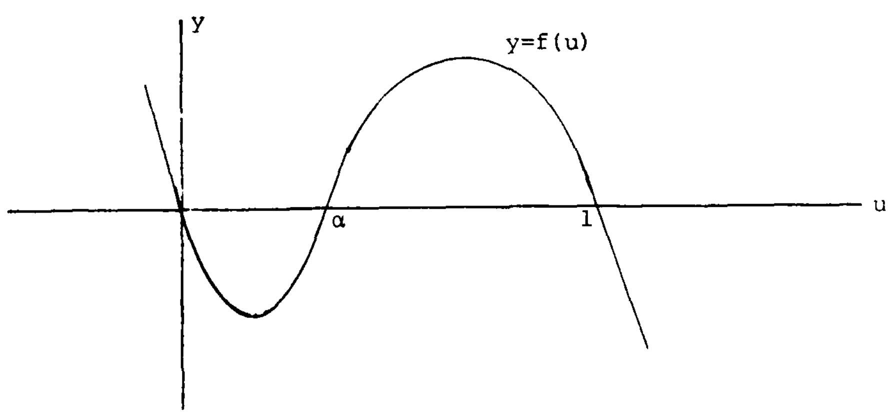

Figure 1.1

where the area of the hump above the x-axis is bigger than that of the hump below. If f satisfies (H2) then the hump above is concave down. It will always be assumed that f satisfies (H1), (H2) will be used in a crucial uniqueness result in chapter 2 and it will be clear what conclusions can be made when (H2) is not satisfied. We shall assume nothing about f outside the interval $[0,1]$ as all our attention will be restricted to data in this set; this is possible as, by the maximum principle (see Chapter 3, section II): if $0 \leq u(x,0) \leq 1$ then $0 \leq u(x,t) \leq 1$ for all $t \geq 0$.

The crucial property of the scalar equation with this $f(u)$ is that it has a unique (up to translation) travelling wave in one space dimension which is stable in a very strong sense. In other words, it has a unique solution of the form $u(x-ct)$ which satisfies the boundary conditions $u(-\infty) = 1$ and $u(+\infty) = 0$ (see Aronson and Weinberger [1] and Fife and McLeod [1]). Fife and McLeod's theorem is the strongest stability statement, it says that the travelling wave is exponentially stable to all initial data which lie above the middle zero at $-\infty$ and below it at $+\infty$ (it is reproduced here as our theorem 5.1).

A standard maximum principle argument implies that $u \equiv 0$ and $u \equiv 1$ are stable (this is the source of its name) to perturbations in the sup-norm, $u \equiv \alpha$ is unstable by the same reasoning. The travelling wave determines the asymptotic behaviour of data that are transitions between the two stable states. Heuristically we could take for initial data characteristic functions of a set; then if $u(x,0)$ is the characteristic function of a half-line $(-\infty,a]$, $u(x,t)$ tends to a translate of the travelling wave as $t \to +\infty$.

In any space dimension we could take $u(x,0)$ to be the characteristic function of a compact set D. If D is large enough and more or less spherical then at the edge of D $u(x,0)$ will not be too unlike a plane front and so one might expect it to propagate in the same fashion as a one-dimensional wave. On the other hand if D were too small we might expect the solution to tend to zero everywhere. Aronson and Weinberger [2] have shown that both of these phenomena occur. They give an integral condition, that will cover some data of the above form if D is small enough, which ensures decay of the corresponding solution. They also give a condition for propagation ($u(x,t) \to 1$ uniformly on compact sets). The latter theorem specialized to the case under consideration is:

Theorem 1.1 (Aronson and Weinberger [2]). With f satisfying (H1) there is an indexed family of functions on $\mathbb{R}^n$ say $\{u_\lambda | \lambda \in \Lambda\}$, each with compact support, so that if, for some λ

$$u(x,0) \geq u_\lambda(x) \quad \text{for all} \quad x \in \mathbb{R}^n$$

and $u(x,t)$ is a solution of (1.3) then $u(x,t) \to 1$ uniformly on compact sets.

They also prove that the propagation has an asymptotic speed which is that of the one-dimensional travelling wave, but we shall not go into this here.

In particular, from the above theorem we know that there is spherical wave propagation for (1.3), it is this pehnomenon that we shall analyze.

IV. SYNOPSIS

From Aronson and Weinberger's results, there is a 'threshold effect', i.e. some solutions decay and some propagate; this suggests that there is an unstable equilibrium solution which demarcates the boundary between these two regimes of behaviour. In chapter 2 we prove the existence of such solutions and under the assumption that f satisfies (H2) we prove it is unique. This solution is obviously unstable and under (H2) we prove that the linearized operator at this equilibrium solution has one positive eigenvalue which, loosely speaking, represents one dimension of decay and propagation.

The remainder of the work is devoted to giving a global picture of this propagation/decay behaviour, using the solution found in chapter 2 as a pivot.

In chapter 3 we compile the necessary information for applying the concepts and methods of dynamical systems. Except for section IV this chapter is all standard material.

In chapter 4 we construct the basic machine that gives out spherical information when we plug in facts about the one-dimensional behaviour. In this chapter we describe the "spherical attractor" when f satisfies (H2), this is the set of permanent solutions that are relevant to this propagation/decay effect.

In chapter 5 we refine the machine of chapter 4 to include a moving co-ordinate frame; this gives finer information about the spherical waves. We also prove the necessary one-dimensional facts in this chapter.

In chapter 6 we sum the work up and translate the results from the language of dynamical systems into terms that are closer to the literature on reaction - diffusion equations.

Notation

(1) For a set A: $\mathrm{Int}(A)$ = interior of A;

$\mathrm{cl}(A)$ = closure of A; A^0 = complement of A.

(2) If $S : Y \times \mathbb{R}^+ \to Y$ is a semiflow and $A \subset Y$, $I \subset \mathbb{R}^+$

$S(t)A = \{S(t)y \mid y \in A\}$; $A \cdot t = S(t)A$;

$A \cdot I = \{y \cdot t \mid y \in A,\ t \in I\}$.

(3) $B = \{u : \mathbb{R}^n \to \mathbb{R}^n \mid u$ is bounded and uniformly continuous$\}$

$\|u\|_B = \|u\|_\infty = \sup_{x \in \mathbb{R}^n} |u(x)|$

(4) If $P = \prod_{i=1}^{n} [a_i, b_i]$ is an invariant rectangle (see chapter 3, section 2), then $M(P) \subset B$ is given by:

$M(P) = \{u(x) = (u(x) = (u_1(x), \ldots, u_m(x)) \mid a_i \le u_i(x) \le b_i$ for every $x \in \mathbb{R}^n\}$ and supplied with the compact-open topology. The dependence on P is usually omitted.

(5) If $u(x,t)$ and $u(x)$ are used together, then $u(x,0) = u(x)$.

Chapter 2

Equilibrium Solutions

I. ONE DIMENSIONAL CASE

In this chapter we shall prove theorems on the existence, uniqueness and spectral properties of nonconstant equilibrium solutions of (1.3). We shall firstly review the well known case of one space dimension.

An equilibrium solution in one space dimension satisfies the equation

$$u_{xx} + f(u) = 0 \quad . \tag{2.1}$$

We are interested in solutions that decay to 0 at $x = \pm\infty$. Transforming (2.1) into a system of ODE's gives

$$\begin{aligned} u' &= v \\ & \qquad\qquad\qquad ' = \frac{d}{dx} \\ v' &= -f(u) \end{aligned} \tag{2.2}$$

which is Hamiltonian and so the phase portrait is easily drawn

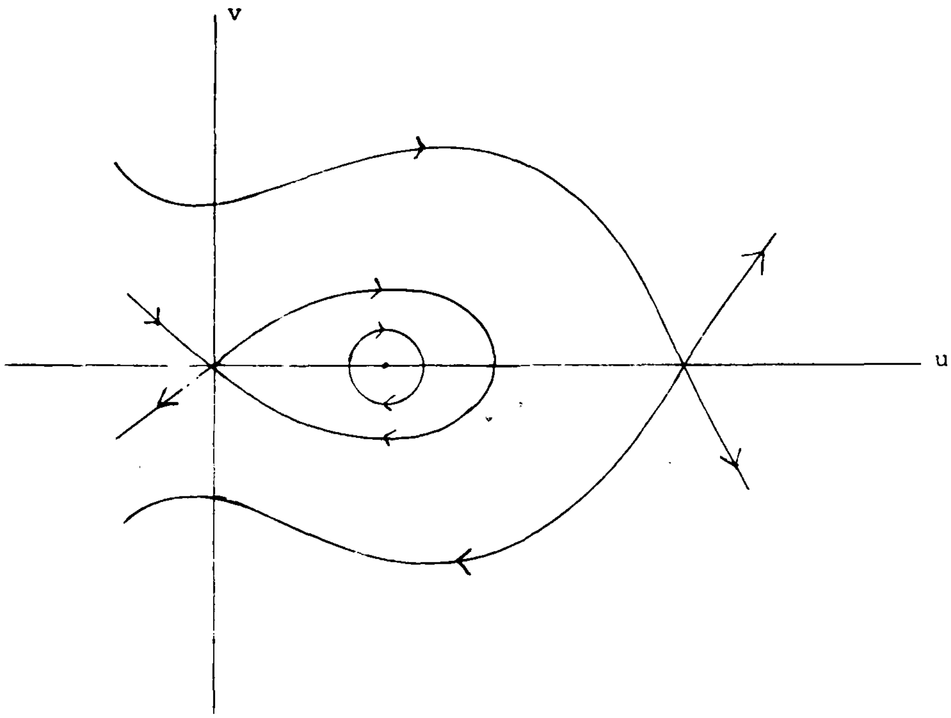

Figure 2.1

From this we can see that there is a unique non-trivial solution (up to translation) $(\bar{u}(x),\bar{v}(x))$ which satisfies the boundary conditions, namely the "body of the fish".

Looking for the eigenvalues of the linearization around this solution leads to the equation

$$
(2.3) \qquad \begin{aligned} u' &= v \\ v' &= (\lambda - f'(\bar{u}))u \end{aligned} \qquad ' = \frac{d}{dx} .
$$

Using the fact that $(\bar{v}(x),\bar{v}'(x))$ is a solution for $\lambda = 0$, by standard comparison arguments it can be shown that there is a unique $\lambda > 0$ for which (2.3) admits a bounded solution.

In the following sections we shall show that the same structure shows up in the higher dimensional case.

II. EXISTENCE

We seek spherically symmetric equilibrium solutions of (1.3); these are solutions of

$$u_{rr} + \frac{n-1}{r} u_r + f(u) = 0 \quad . \tag{2.4}$$

Theorem 2.1. For fixed n and f satisfying (H1) there is at least one bounded non-constant solution of (2.4) that satisfies

$$u(r = +\infty) = 0 \quad .$$

Proof: Transforming (2.4) into a system, we get

$$\begin{aligned} u' &= v \\ v' &= -\frac{n-1}{r} v - f(u) \quad . \end{aligned} \qquad ' = \frac{d}{dr} \tag{2.5}$$

Roughly speaking, when $r = +\infty$ we have the equilibrium equation for the one-dimensional problem. In order to make this precise we perform the transformation $\rho = \frac{r}{r+1}$

$$\begin{aligned} u' &= v \\ v' &= -\frac{(n-1)(1-\rho)}{\rho} v - f(u) \\ \rho' &= (1-\rho)^2 \quad . \end{aligned} \qquad ' = \frac{d}{dr} \tag{2.6}$$

There is a singular surface at $\rho = 0$, but otherwise this equation is well defined on $\mathbb{R}^2 \times [0,1]$. The phase portrait in the section $\rho = 1$ is easily seen to be that of Figure 2.1.

Since the solution must be regular at the origin, we seek a solution of (2.6) that satisfies the boundary conditions

(2.7) $$v(0) = 0$$

(2.8) $$u(\infty) = 0 \quad .$$

(2.8) means that the purported solution must lie in the set W given by:

$$W = \{(u,v,\rho) \mid (u(r),v(r),\rho(r)) \to (0,0,1) \quad \text{as} \quad r \to +\infty \quad \text{and} \quad (u(0),v(0),\rho(0)) = (u,v,\rho)\} \quad .$$

By standard results, see for instance Kelley [1], there is a C^1 local center-stable manifold, call it W^{cs}_{loc}, at $(0,0,1)$ whose tangent space is generated by ζ_s and $(0,0,-1)$ where ζ_s is any vector in u,v space tangent to the stable manifold of $(0,0)$ in Figure 2.1.

Now let $T(r)$ be the solution operator of (2.6), for all r $T(r)W^{cs}_{loc} \subset W$. Define $W^{cs} = \bigcup\limits_{r\leq 0} T(r)W^{cs}_{loc}$, then $W^{cs} \subset W$. Since $W^{cs} \cap \{\rho=1\}$ contains points in the positive quadrant of the u,v plane, see Figure 2.1, if r is sufficiently large negative $T(r)W^{cs}_{loc}$ contains these points. Therefore, since W^{cs}_{loc} is a manifold and $T(r)$ is a diffeomorphism, $T(r)W^{cs}_{loc} \cap \{\rho = \bar{\rho} < 1\}$ intersects the positive quadrant of $\rho = \bar{\rho}$, for $\bar{\rho}$ close to 1.

The points $W \cap \{u, v > 0\}$ stay in $\{u, v > 0\}$ under application of $T(r)$ for $r < 0$. To see this we must consider the Hamiltonian of the one-dimensional equation

$$H(u,v) = \frac{v^2}{2} + \int_0^u f(s)ds \quad .$$

We compute $\dot{H}$ along orbits of (2.6)

$$\dot{H} = -\frac{(n-1)(1-\rho)}{\rho} v^2$$

so if $\rho \neq 1$ and $v \neq 0$ then $\dot{H} < 0$. For the one-dimensional equation the energy curves are the solution curves. If $\rho \neq 1$, the solutions of (2.6) cross these curves with decreasing energy. We borrow the two curves C_1 and C_2 in Figure 2.2 below from Figure 2.1 (they are part of the energy surface $H = 0$) and setting $C_3 = \{v=0,\ \gamma \leq u \leq 1\}$ and $C_4 = \{u=1,\ v > 0\}$, it is easy to see that the vector field in the $\rho = \bar{\rho}$ slice is as depicted on $C = C_1 \cup C_2 \cup C_3 \cup C_4$.

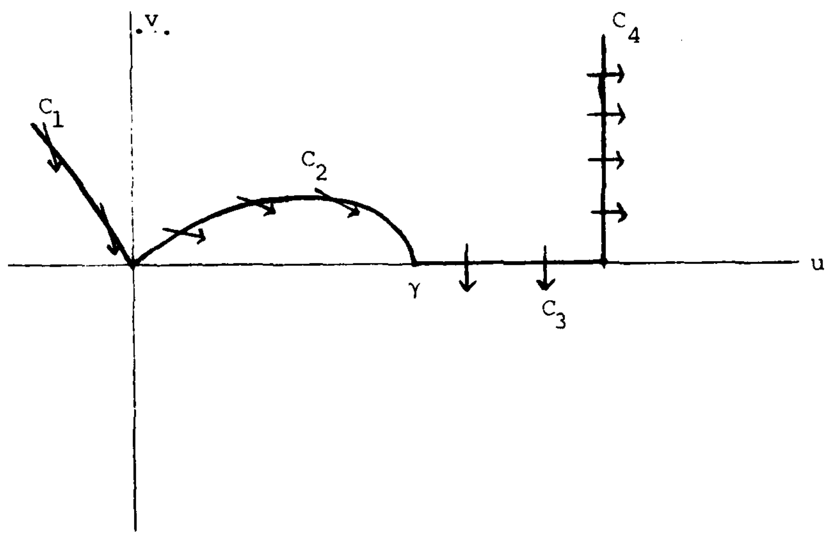

Figure 2.2

so the region bounded by C in the upper half-plane is negatively invariant. Since $\dot{H} < 0$, W^{cs} must lie outside $H = 0$ and so it

crosses C on C_3 for $\bar{\rho}$ close to 1. Therefore W^{cs} intersects the positive quadrant for every $\bar{\rho} > 0$.

Let $\varepsilon > 0$ be a fixed small number and define a set D bounded by the four lines

$$u = \alpha,\ -\varepsilon \le v \le \varepsilon;\ v = \pm\varepsilon,\ \alpha \le u \le 1;\ u = 1,\ -\varepsilon \le v \le \varepsilon\ .$$

From the above $W^{cs} \cap D \cap \{\rho=\bar{\rho}\} \neq \phi$ for every $\bar{\rho} > 0$. We want to show there is a solution in W^{cs} which is in D at some $\bar{\rho}$ and stays in for all $0 \le \rho \le \bar{\rho}$, renormalizing the independent variable then gives one that satisfies (2.7).

Let $D_U = \partial D \cap \{v>0\}$ and $D_L = \partial D \cap \{v<0\}$, then for every $\bar{\rho}$, $W^{cs} \cap \{\rho=\bar{\rho}\} \cap D_U \neq \phi$ and $W^{cs} \cap \{\rho=\bar{\rho}\} \cap D_L \neq \phi$.

If $\bar{\rho}$ is small enough the vector field on ∂D is as shown below

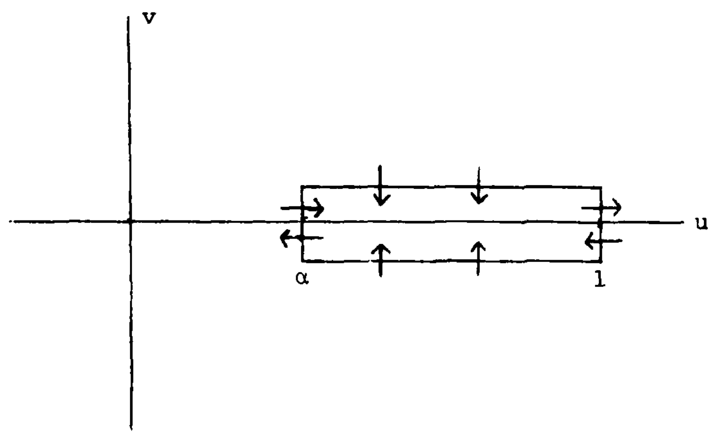

Figure 2.3

For this $\bar{\rho}$, let F be a compact curve in $W^{cs} \cap \{\rho=\bar{\rho}\}$ that intersects D_U and D_L, such a curve clearly exists. Let $F_U = \{y \in F \mid$ for some $r < 0$ $T(r)y \notin D$ but $T(r)y \in \{v>0\}\}$ and F_L be the corresponding set with $v < 0$. F_U and F_L are obviously both open and from the picture in Figure 2.3 are both non-empty but then

$$F \neq F_U \cup F_L$$

so there exists $\gamma \in F$ that stays in B for all ρ such that $0 \leq \rho \leq \bar{\rho}$. This completes the proof of Theorem 2.1.

This solution, $\bar{u}(r)$, found in theorem 1 satisfies the following two properties

(A) $\bar{u}'(r) < 0$ for all $r \in (0,+\infty)$.

(B) $\gamma < \bar{u}(0) < 1$ where $\gamma : \alpha < \gamma < 1$ is determined by the condition $\int_0^{\gamma} f(s)ds = 0$.

Property (A) is clear from the fact that the region described in Figure 2.2 is negatively invariant. Since W^{cs} lies outside $H = 0$ it is clear that $\bar{u}(0) > \gamma$. To see that $\bar{u}(0) < 1$ we apply the maximum principle. Since $u \equiv 1$ satisfies (2.4), by an application of the mean value theorem $w = 1-\bar{u}$ satisfies

$$\Delta w + c(x)w = 0 \tag{2.9}$$

for some bounded function $c(x)$. But by (A) and the obvious fact that $\bar{u}(0) \leq 1$ we see that $1-\bar{u} \geq 0$; but then, by the maximum principle, see Protter and Weinberger [1], $1-\bar{u}$ cannot have an interior minimum of zero which it would if $\bar{u}(0) = 1$.

III. SPECTRAL ANALYSIS

In this section we consider the eigenvalue problem for the linearized operator around a solution $\bar{u}$ of the form found in theorem ... Specifically we want to know for which λ's the equation

$$\Delta u + (f'(\bar{u}) - \lambda)u = 0 \tag{2.10}$$

admits a bounded solution. In particular we want to know for which $\lambda \geq 0$ (2.10) admits a bounded solution.

It follows from standard results, see e.g. Reed and Simon [1] Vol. IV, that any positive eigenvalue must correspond to a spherically symmetric eigenfunction. This comes from the fact that 0 is an eigenvalue with an associated n-dimensional eigenspace generated by $\{\bar{u}_{x_1}, \ldots, \bar{u}_{x_n}\}$ (this is the translation eigenvalue) and each of these has no zero except at zero. The rest of the spectrum is bounded away from zero to its left.

So we look for solutions of the equation, with $\lambda \geq 0$,

$$u_{rr} + \frac{n-1}{r} u_r + (f'(\bar{u}) - \lambda)u = 0 \quad . \tag{2.11}$$

<u>Theorem 2.2</u>. If f satisfies (H1) and (H2), there is only one $\lambda \geq 0$ for which (2.11) admits a bounded solution and that λ is positive.

<u>Proof</u>: Converting (2.11) to a system

$$\begin{aligned} u' &= v \\ v' &= -\frac{n-1}{r} v + (\lambda - f'(\bar{u}))u \quad . \end{aligned} \tag{2.12}$$

The associated angular equation is

$$\theta' = -\frac{n-1}{r} \sin\theta \cos\theta + (\lambda - f'(\bar{u}))\cos^2\theta - \sin^2\theta \tag{2.13}$$

where

$$\theta = \arctan(\frac{v}{u}) \quad .$$

We only have to worry about solutions of (2.13).

Roughly speaking, at $r = +\infty$ (2.13) is the angular equation for the system

$$\text{(2.14)} \qquad \begin{aligned} u' &= v \\ v' &= (\lambda - f'(\bar{u}))u \quad . \end{aligned}$$

This equation has stable and unstable subspaces which are lines of slope $-\sqrt{\lambda - f'(0)}$ and $+\sqrt{\lambda - f'(0)}$ respectively. Let $\bar{\theta}$ be the angle (in the fourth quadrant) that this stable subspace makes with the u-axis. It is not hard to see that if (2.12) has a bounded solution, the corresponding solution $\theta(r)$ to (2.13) must satisfy

$$\text{(2.15)} \qquad \theta_\lambda(\infty) = \bar{\theta}_\lambda \pmod{\pi} \quad .$$

It is also standard that there is a unique solution, $\bar{\theta}_\lambda(r) \pmod{\pi}$, of (2.13) that satisfies (2.15).

The proof requires a study of $\bar{\theta}_\lambda(r)$ and its limit as r tends to 0. The solution of the full equation remains bounded only if $\bar{\theta}_\lambda(0) = 0 \pmod{\pi}$, so the question is for what λ's can we have

$$\text{(2.16)} \qquad \bar{\theta}_\lambda(0) = m\pi \qquad \text{for some } m \ ?$$

We will split the proof up into three parts but firstly we make some preliminary observations. We must have $m \geq 0$ as when $\theta = -\pi/2$, $\theta' = -1$. The λ's which work for m are decreasing in m, that is, if λ_i works for m_i $i = 1,2$ and $m_1 < m_2$ then $\lambda_2 < \lambda_1$. This follows from the fact that if $\lambda > \hat{\lambda}$, and $\theta_\lambda(r)$, $\theta_{\hat{\lambda}}(r)$ are solutions of (2.13) with λ and $\hat{\lambda}$ respectively then if $\theta_\lambda(R) \leq \theta_{\hat{\lambda}}(R)$ (we may even have $R = +\infty$ here) then $\theta_\lambda(r) \leq \theta_{\hat{\lambda}}(r)$ for all $r \leq R$. This latter fact is proved by a comparison between the λ and $\hat{\lambda}$ equations.

From this we know that the set of λ's giving an affirmative answer to (2.16) are bounded above and the associated m's are increasing with decreasing λ. The rest of the proof is split up into proving the following three parts.

(1) For a given m there is at most one λ which satisfies (2.16).

(2) There is a $\lambda > 0$ for which $\bar{\theta}_\lambda(0) = 0$.

(3) $0 < \bar{\theta}_0(0) < \pi$.

Once these are proved the proof is complete since, by (3) and the decreasing property, if (2.16) can be solved for $m = 1$, the associated λ must be negative and therefore so must all other λ's be negative associated to any $m \geq 1$. The only possible $\lambda \geq 0$ must therefore correspond to $m = 0$, from (1) there is a unique such λ and (2) shows that it exists. The hypothesis (H2) is used in the proof of (3) and this is perhaps the heart of the theorem.

<u>Proof of (1)</u>: Suppose λ satisfies (2.16) for some m, we firstly show that $\bar{\theta}_\lambda(r)$ is the only solution (mod π) of (2.13) (with λ fixed at this value) which satisfies (2.16).

Suppose it were not, then we would have two solutions (u_1,v_1) and (u_2,v_2) of (2.12) with the property that $v_1 - v_2 \to 0$ as $r \to 0$, and also that $u = u_1 - u_2$, $v = v_1 - v_2$ is a solution of

(2.12). The angular and radial equations of (2.12) are given by (2.13) and

(2.17) $$\rho' = -\rho(\sin\theta\cos\theta(1+\lambda - f'(\bar{u})) - \frac{n-1}{r}\sin^2\theta)$$

where

$$\rho = (u^2 + v^2)^{1/2} .$$

From (2.13) if r is small enough the shaded cone below is negatively invariant (down to $r = 0$)

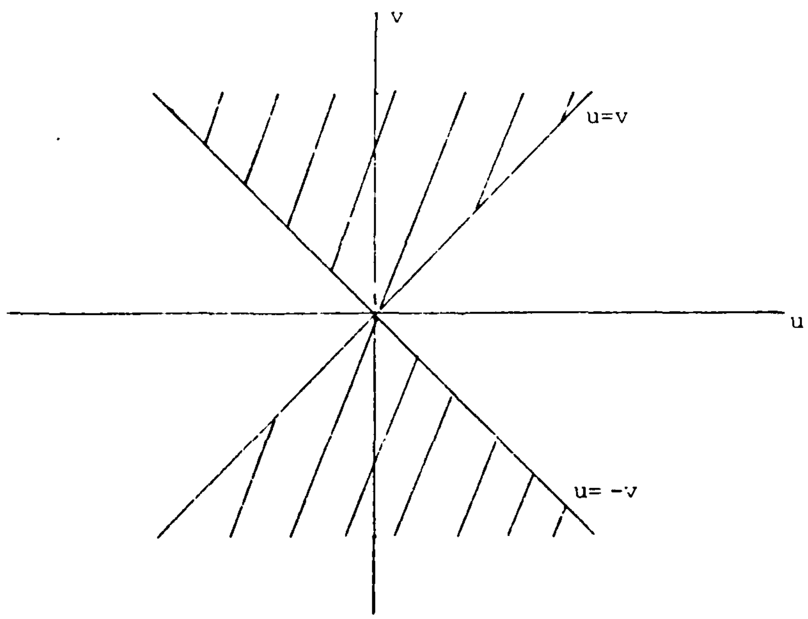

Figure 2.4

If a solution remained in this cone as $r \to 0$, from (2.17) it would have either $\rho \equiv 0$ or $\rho \to +\infty$ as $r \to 0$. But by assumption we could choose (u_1,v_1) and (u_2,v_2) so that (u,v) lies in this cone for some small r. So if $v_1 - v_2 \to 0$ we would have to have $\rho \equiv 0$ and so $(u_1,v_1) = (u_2,v_2)$.

Assume that $\lambda_1 < \lambda_2$ then there must be an [illegible] $\bar{\theta}_{\lambda_1}(R) > \bar{\theta}_{\lambda_2}(R)$. But then there exists a solution t[illegible] $\lambda = \lambda_2$, call it $\theta_{\lambda_2}(r)$, so that

$$\bar{\theta}_{\lambda_1}(R) > \theta_{\lambda_2}(R) > \bar{\theta}_{\lambda_2}(R)$$

but then for all $r \leq R$ $\bar{\theta}_{\lambda_1}(r) > \theta_{\lambda_2}(r)$ and since [illegible] unique solution of (2.13) with $\lambda = \lambda_2$ that satisfies [illegible], $\theta_{\lambda_2}(0) > m\pi$ and so $\bar{\theta}_{\lambda_1}(0) > m\pi$.

<u>Proof of (2)</u>: That such a λ exists for $m = 0$ follows from a straightforward shooting argument using the two facts: (i) if [illegible] then $\bar{\theta}_\lambda(0) \leq \bar{\theta}_\lambda$ and (ii) if $\lambda = 0$, $\bar{\theta}_\lambda(0) \geq \pi/2$.

(i) follows from the fact that if $\lambda >> 0$, $\theta = \bar{\theta}_\lambda$ impl[illegible] (from (2.13)) and so $\bar{\theta}_\lambda(r) \leq \bar{\theta}$ for all $r \geq 0$. For (ii) (2.1[illegible] be compared, when $\lambda = 0$, to the angular equation of

$$\text{(2.18)} \qquad \begin{aligned} u' &= v \\ v' &= -\frac{n-1}{r}v - f'(\bar{u})u + \frac{1}{r^2}u \end{aligned}$$

which has a known solution, namely $\bar{u}'(r)$. It is easy to see that th[illegible] forces $\bar{\theta}_0(0) \geq \pi/2$.

<u>Proof of (3)</u>: For the remainder of this section we shall be discussi[illegible] (2.13) with $\lambda = 0$ and so, for the sake of notation, will drop the subscript 0 on $\bar{\theta}_0$ and $\bar{\theta}_0(r)$. With $\lambda = 0$, (2.13) is

$$\text{(2.19)} \qquad \theta' = -\frac{n-1}{r}\sin\theta\cos\theta - f'(\bar{u})\cos^2\theta - \sin^2\theta \quad .$$

This is also the angular equation of the tangent (or first variational) equation of (2.12) about the solution $(\bar{u}(r),\bar{v}(r))$ and $\bar{\theta}$ is the angle of the stable manifold to $(0,0)$ in Figure 2.1. Consequently $\bar{\theta}(r)$ is the angle of the tangent line to the manifold $W_R = W \cap \{\rho = \frac{R}{R+1}\}$ at $(\bar{u}(R),\bar{v}(R))$. We actually have not shown that this is a manifold yet, a proof of this is in section III, but we do know it contains a manifold, namely $W^{cs} \cap \{\rho = \frac{R}{R+1}\}$ and for the following arguments this suffices.

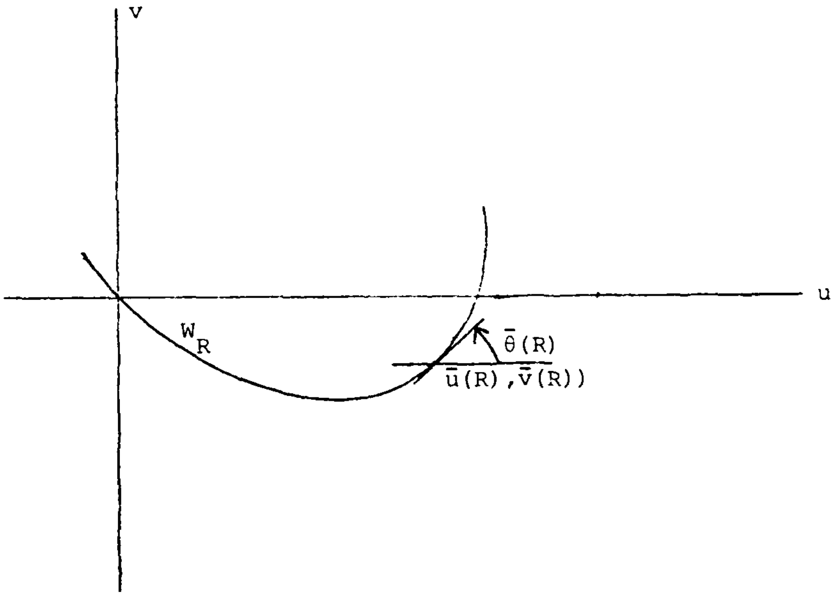

Figure 2.5

We must prove that $\bar{\theta}(0) < \pi$, we do this in two steps, it is in step 2 that the hypothesis (H2) is used.

Remark: (3) is about $\bar{\theta}(r)$ which is a solution of (2.13) associated to a $\bar{u}(r)$ which satisfies (2.4), (2.8) and its derivative satisfies (2.7). But in fact we only need that (2.4) and (2.8) are satisfied to obtain that $\bar{\theta}(0) < \pi$ or $\bar{v}(0) > 0$. We shall use this fact in the next section.

Step 1: We prove that if $\bar{\theta}(R) = 0$ then $\bar{u}(R) \geq \alpha$. Define the set $W_R^{cs} = W^{cs} \cap \{\rho = \frac{R}{R+1}\}$, then define $D_{R_0} = \{(u,R^1) | W_{R^1}^{cs}$ has a horizontal tangent line opposite u and $R_0 \leq R^1 \leq +\infty\}$. If we assume $\bar{u}(R) < \alpha$ then since also D_R is closed, there is an R such that $(u,R) \in D_{R_0}$ for some u and if $(u_1,R_1) \in D_{R_0}$ $u_1 \geq u$ and if $u_1 = u$ then $R_1 \leq R$.

So W_R^{cs} must cross energy levels of H in increasing value opposite u_1 close to but less than u. See Figure 2.6.

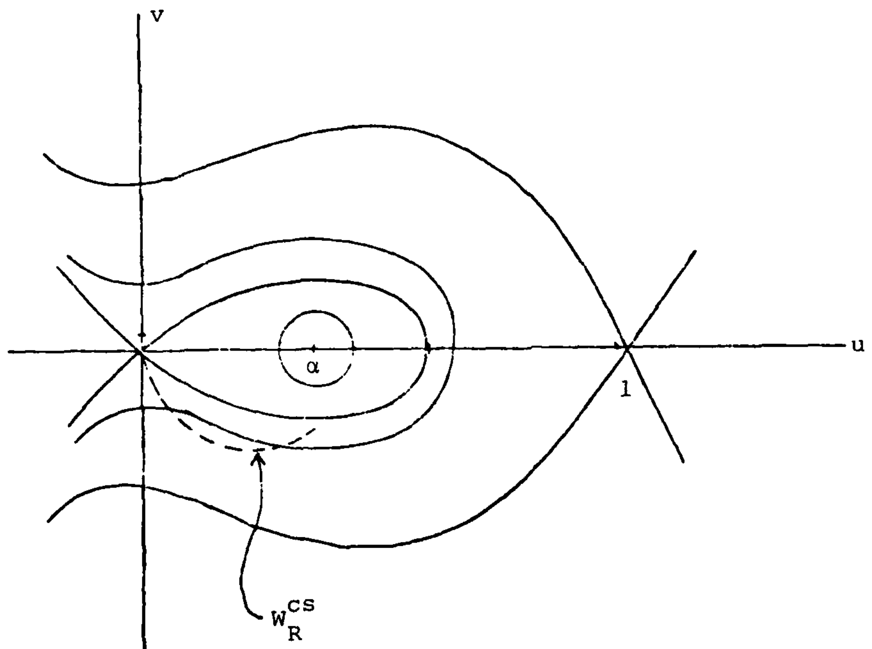

Figure 2.6

So then there are points (u_1,v_1) and (u_2,v_2) on W_R^{cs} so that if $(u_i(r),v_i(r))$, $i = 1,2$ satisfy $(u_i(R),v_i(R)) = (u_i,v_i)$, we have

$$H(u_1,v_1) > H(u_2,v_2)$$

and yet $(u_i(r),v_i(r)) \to 0$ as $r \to +\infty$. Recall that

$$\dot{H}(u_i(r),v_i(r)) = -\frac{(n-1)(v_i(r))^2}{r}$$

fron the definition of R and the fact that $u_1 < u_2$ we have $v_1(r) < v_2(r)$ and so $\dot{H}(u_1(r),v_1(r)) < \dot{H}(u_2(r),v_2(r))$ but then $H(u_1(\infty),v_1(\infty)) > H(u_2(\infty),v_2(\infty))$ which is a contradiction and so $\bar{u}(R) \geq \alpha$.

Step 2: If $\bar{\theta}(R) = \pi$ then $\bar{v}(R) > 0$ $(R \geq 0)$.

Consider the angular variation of the full nonlinear equation (2.5) about the point $(\alpha,0)$ in the u,v-plane,

$$\psi = \arctan\left(\frac{\bar{v}}{\bar{u}-\alpha}\right) .$$

This quantity satisfies

$$\psi' = -\frac{n-1}{r}\sin\psi\cos\psi - f'(\bar{u})\cos^2\psi - \sin^2\psi + g(r) \tag{2.20}$$

where $g(r)$ is given by the expression

$$\begin{aligned} g(r) &= -\frac{f(\bar{u})(\bar{u}-\alpha)}{\bar{v}^2+(\bar{u}-\alpha)^2} + f'(\bar{u})\cos^2\psi \\ &= -\frac{f(\bar{u})(\bar{u}-\alpha) + f'(\bar{u})(\bar{u}-\alpha)^2}{\bar{v}^2+(\bar{u}-\alpha)^2} \\ &= \frac{\bar{u}-\alpha}{\bar{v}^2+(\bar{u}-\alpha)^2}\left(f'(\bar{u})(\bar{u}-\alpha) - f(\bar{u})\right) . \end{aligned}$$

Now if $f''(\bar{u}) \leq 0$ then $f'(\bar{u})(\bar{u}-\alpha) - f(\bar{u}) \leq 0$ and so if $\bar{u}$ [illegible] have

$$g(r) \leq 0 \quad . \tag{2.21}$$

Suppose $\bar{\theta}(R_0) = 0$, then $\bar{u}(R_0) \geq \alpha$ and so $\bar{u}(r)$ [illegible] (unless $\bar{v}(r)$ becomes fairly positive, which case does not interest us). By (H2) if $\bar{u} \geq \alpha, f''(\bar{u}) \leq 0$ and so if $r \leq R_0$ (2.21) [illegible]. Since $\bar{u}(R_0) \geq \alpha$, $\psi(R_0) \geq -\pi/2$.

$\bar{\theta}(r) - \pi$ satisfies (2.19) and $\psi(r)$ satisfies (2.20) and since $\bar{\theta}(R_0) - \pi \leq \psi(R_0)$ we must have $\theta(r) - \pi \leq \psi(r)$ for all r such that $0 \leq r \leq R_0$. So if $\bar{\theta}(R) = \pi$, $\psi(R) \geq 0$ and so $\bar{v}(R_1) \geq 0$. If $R_1 \neq$ [illegible] we have strict inequality and since $\bar{\theta}(R_0) - \pi < \psi(R_0)$, as before (in the proof of (1)) $\psi(r)$ can be compared to another solution of (2.19) and so $\psi(0) > 0$ in this case also. In either case $\bar{v}(R)$ [illegible] and the proof of step 2 is complete.

To complete the proof of (3) notice that we have shown that if $\bar{\theta}(R) = \pi$ then $\bar{v}(R) > 0$, but this is impossible for any solution of theorem 2.1 as according to property (A) $\bar{v}(r) \leq 0$. So clearly we must always have $\bar{\theta}(0) < \pi$.

IV. UNIQUENESS

From the fact that the region bounded by C in Figure 2.2 is negatively invariant the only solutions to (2.5) that satisfy the boundary condition $u(+\infty) = 0$ and stay between 0 and 1 must satisfy properties (A) and (B) at the end of section I.

W^{cs} is obtained by iterating W^{cs}_{loc} in backward r and is consequently a manifold, but we do not know yet that everything which tends to $(0,0,1)$ as $r \to +\infty$ is in W^{cs}, i.e. intersects W^{cs}_{loc}.

In a neighbourhood of $(0,0,1)$, the equation, by an affine change of co-ordinates, has the form:

$$y' = Ay + g(y) \qquad y = (y_1, y_2, y_3) \tag{2.22}$$

where

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda_1 > 0, \quad \lambda_2 < 0$$

and

$$g'(0) = 0 \quad .$$

Consider the cone C_0 given by

$$C_0 = \{y = (y_1, y_2, y_3) \mid |y_1| \geq |(y_2, y_3)|\} \quad ,$$

then let $C_y = y + C_0$. Let $\varphi(r)$ be the solution of (2.22) such that $y = \varphi(0)$, $\psi(r)$ be that with $z = \psi(0)$.

<u>Lemma 2.1</u>. There exists a neighbourhood U of $(0,0,0)$ so that if $z, y \in U$ and $z \in C_y$ then $|\pi_1(\psi(r) - \varphi(r))|$ is increasing so long as $\psi(r)$ and $\varphi(r)$ are both in U.

Notice that $(0,0,1)$ of (2.5) is transformed into $(0,0,0)$ of (2.22). For the sake of notation call the image of W^{cs}_{loc} also W^{cs}_{loc}. Take $z \in U$ but $z \notin W^{cs}_{loc}$, clearly there is a $y \in U \cap W^{cs}_{loc}$ so that $z \in C_y$, but since $\varphi(r) \to 0$ as $r \to +\infty$, we cannot have $\psi(r) \to 0$

without it leaving U first, by the lemma. Transforming this statement back to the original variables gives us that $W = W^{cs}$ as desired.

Proof of Lemma 2.1.

Take U to be a ball of radius $\varepsilon > 0$, so small that

$$|g(w) - g(\bar{w})| \leq \delta|w-\bar{w}|$$

for $w,\bar{w} \in U$. If $P(r)$ is the evolution operator of (2.22) then δ can be chosen so that

$$P(r)\ (C_y \cap U) \subset C_{\varphi(r)}$$

for $y \in U$ so long as $\varphi(r) \in U$; i.e. these cones are invariant under the flow. Suppose $z \in C_y$ then $z-y \in C_0$ and the associated solution $\psi(r) - \varphi(r)$ satisfies the equation

$$(\psi-\varphi)' = A(\psi-\varphi) + g(\psi) - g(\varphi) \tag{2.23}$$

and since $\lambda_1 > 0$, it is clear that if δ is small the function on the right points inward on $\partial C_0 \cap U$.

The solution $\varphi(r)$ is given by the variation of constants formula.

$$\varphi(r) = e^{Ar}y + \int_0^r e^{A(r-s)}g(\varphi(s))ds$$

and so,

$$\pi_1(\psi(r) - \varphi(r)) = e^{\lambda_1 r}\pi_1(z-y) + \int_0^r e^{\lambda_1(r-s)}\pi_1(g(\psi(s)) - g(\varphi(s)))ds$$

Because $\psi(s) - \varphi(s) \in C_0$ there must exist a $k_1 > 0$ so that

$$|\pi_1(g(\psi(s)) - g(\varphi(s)))| \leq k_1\delta|\pi_1(\psi(s) - \varphi(s))|$$

but then

$$|\pi_1(\psi(r) - \varphi(r))| \geq e^{\lambda_1 r}|\pi_1(z-y)| - \int_0^r e^{\lambda_1(r-s)} k_1\delta|\pi_1(\psi(s) - \varphi(s))|ds \quad .$$

Now if we assume that $|\pi_1(\psi(s) - \varphi(s))| \leq k_2 e^{\lambda s}|\pi_1(z-y)|$ where $\lambda < \lambda_1$ (otherwise we would be done) we get

$$\begin{aligned} |\pi_1(\psi(r) - \varphi(r))| &\geq e^{\lambda_1 r}(|\pi_1(z-y)| - k\delta|\pi_1(z-y)|) \\ &= e^{\lambda_1 r}(1-k\delta)|\pi_1(z-y)| \quad . \end{aligned}$$

If δ is chosen small enough the lemma follows.

We can now state and prove the uniqueness theorem.

Theorem 3.3. If f satisfies (H1) and (H2) then the nonconstant solution of Theorem 3.1 is the unique one between 0 and 1.

Proof. We firstly fix some notation. From lemma 3.1 we know that for each R, W_R is a one-dimensional curve, and so $W_R \setminus (u,v)$ has two connected components call these $W_R^0(u,v)$ and $W_R^1(u,v)$, where they are determined by the requirement $0 \in W_R^0(u,v)$. Let $T_R(u,v)$ be the tangent line to W_R at (u,v) and $T_R^0(u,v)$ be the half-line of $T_R(u,v)\setminus\{0\}$ that is tangent to $W_R^0(u,v)$, define $T_R^1(u,v)$ similarly. It is clear that the evolution map of the equation maps $W_R^i(u(R),v(R))$ to $W_r^i(u(r),v(r))$ $i = 0,1$ if $(u(r),v(r))$ is a solution. A similar statement is true for the action of the tangent equation on T_R^i.

Now let $(\bar{u}(r),\bar{v}(r))$ be a solution of (2.5) lying in [illegible] $\bar{v}(r) \leq 0$ for $r \leq R$, some R, and $\bar{\theta}(r)$ be the [illegible] solution of the angular variational equation (2.19). If [illegible] then $T_R^1(\bar{u}(R),\bar{v}(R))$ is the upper half of $T_R(\bar{u}(R),\bar{v}(R))$. [illegible] because $T_\infty^1(0,0)$ points down and if $\bar{\theta}(R) > 0$ it must [illegible] the horizontal at least once if $T_R^1(\bar{u}(R),\bar{v}(R))$ were pointing [illegible] would have passed through the horizontal twice. From the proof [illegible] in Theorem 2.2 it cannot rotate through more than an angle of [illegible] but passing through the horizontal twice would require this.

Now suppose $(u_1(r),v_1(r))$ and $(u_2(r),v_2(r))$ are two solutions of the problem. Let $u_1(0) < u_2(0)$ and suppose there is no solution $u(r)$ with $u_1(0) < u(0) < u_2(0)$ nor any with $u(0) < u_1(0)$, then either

$$(u_1(R),v_1(R)) \in W_R^1(u_2(R),v_2(R)) \tag{2.24}$$

or

$$(u_1(R),v_1(R)) \in W_R^0(u_2(R),v_2(R)) \tag{2.25}$$

for all R sufficiently small. (2.25) is impossible by comparison with equation (2.18). If (2.24) holds then since $T(u_2(R),v_2(R))$ is bounded away from the horizontal and $T_R^1(u_2(R),v_2(R))$ points up, for R sufficiently small there is a point $(u,v) \in W_R^1(u_2(R),v_2(R))$ with $v > 0$ and $u > u_1(R)$. Buth then since $T_R^0(u_1(R),v_1(R))$ is pointing down there must exist another point (u_3,v_3) with the properties (1) $v_3 < 0$ and (2) $T^1(u_3,v_3)$ is the lower half of $T(u_3,v_3)$ which is a contradiction to the Remark on p. 23.

Remark: We could have proved uniqueness by reference to the full PDE (1.3), the spectral properties of Theorem 2.2 mean that if there was more than one solution their Morse indices would not have added up correctly. What we have done is a geometric version of the same idea and is done without departure from the ODE.

Chapter 3

Generalities

I. LOCAL EXISTENCE AND UNIQUENESS

In this chapter we compile the appropriate generalities about systems of reaction - diffusion equations, everything is standard material except for section IV. Apart from some reversion to the scalar case in section II we will consider the system of equations (1.1).

One of the basic spaces (1.1) can be solved in is:

$$B = \{u : \mathbb{R}^n \to \mathbb{R}^m \mid u \text{ is bounded and uniformly continuous}\} ,$$

where

$$\|u\|_B = \|u\|_\infty .$$

To be more precise, given $u(x) \in B$ there is a unique solution $u(x,t)$ which is a distributional solution of (1.1) with $u(x,t) \in B$ for sufficiently small t and $u(x,0) = u(x)$.

The fundamental solution of the heat equation is

$$K(x,t) = (4\pi t)^{-n/2} \exp[-|x|^2/4t] . \tag{3.1}$$

It is a standard matter to show that $u(x,t) \in C([0,T],B)$ (the continuous functions from $[0,T]$ into B) is a solution to (1.1) with $u(x,0) = u(x)$ if and only if each component $u_i(x,t)$ satisfies the integral equation

$$u_i(x,t) = K(x,t)*u_i(x,t) + \int_0^t K(x,t-s)*f_i(u(x,s))ds \tag{3.2}$$

where $*$ denotes convolution with respect to x (it is here that the boundedness of functions in B enters).

If (3.2) is considered as a system of equations $(i = 1,\ldots,m)$, the right hand side is a mapping on $C([0,T],B)$. A contraction mapping argument then shows that for small enough T (3.2) can be solved for any $u(x) \in B$, and furthermore T only depends on $\|u\|_B$. An application of the Gronwall Inequality shows that the solution is unique and continuous jointly in time and initial data. For details of these arguments see, for instance, Rauch and Smoller [1].

To sum this up, the solutions define a local semiflow on B, i.e. if we define

$$S(t)u(x) = u(x,t)$$

then S is a local semiflow on B.

II. COMPARISON PRINCIPLES

To obtain qualitative information about solutions of a parabolic equation the main tool is usually the maximum principle. We collect here some of the corollaries of the maximum principle that give order relations between the solutions of a reaction - diffusion equation.

The first three principles refer only to the scalar equation, so we assume $u \in \mathbb{R}$. Any functions $u(x,t)$ mentioned are continuous on some interval $[0,T]$ into B, in the obvious fashion. Since smooth functions are dense in B, we can assume $u(x,t)$ is smooth when proving inequalities such as those below, as the general case then follows by taking limits.

Principle 3.1. Suppose $f(0) = 0$ and

$$\frac{\partial u}{\partial t} - \Delta u - f(u) \leq 0 \tag{3.3}$$

in some domain $\bar{\Omega} \times [0,T]$ where $\Omega \subset \mathbb{R}^n$. Assume further

$$u(x,t) \leq 0 \quad \text{on} \quad \partial\Omega \times [0,T] \tag{3.4}$$

$$u(x,0) \leq 0 \tag{3.5}$$

then

$$u(x,t) \leq 0 \quad \text{on} \quad \bar{\Omega} \times [0,T] \ . \tag{3.6}$$

Principle 3.2. Suppose

$$\frac{\partial u}{\partial t} - \Delta u - f(u) \geq \frac{\partial v}{\partial t} - \Delta v - f(v) \tag{3.7}$$

in some domain $\bar{\Omega} \times [0,T]$ where $\Omega \subset \mathbb{R}^n$. Assume further

$$u(x,t) \geq v(x,t) \quad \text{on} \quad \partial\Omega \times [0,T] \tag{3.8}$$

$$u(x,0) \geq v(x,0) \tag{3.9}$$

then

$$u(x,t) \geq v(x,t) \quad \text{on} \quad \bar{\Omega} \times [0,T] \ . \tag{3.10}$$

Proof of Principles 3.1 and 3.2. 3.1 clearly follows from 3.2 by setting $v(x,t) = 0$ and reversing the inequalities. 3.2 follows from the standard linear maximum principle since, by the mean value theorem $f(u) - f(v) = f'(\eta)(u-v)$ and so (3.7) becomes

$$\frac{\partial}{\partial t}(u-v) - \Delta(u-v) - f'(\eta)(u-v) \geq 0 \ .$$

This is the standard argument, see Protter and Weinberger [1].

A function $u(x,t)$ that satisfies (3.10), for any solutions $v(x,t)$ that satisfy (3.8) and (3.9) is called a subsolution. It is obvious that if $\{u_1, \ldots, u_n\}$ are subsolutions then

$$u(x,t) = \max_{i=1,\ldots,n} u_i(x,t)$$

is also a subsolution. The analogous statements hold for supersolutions.

Principle 3.3. Suppose we have k solutions of the inequality

$$\Delta u + f(u) \geq 0 \tag{3.11}$$

say $u_1(x), \ldots, u_k(x)$ on $\Omega_1, \ldots, \Omega_k$. Suppose $\mathbb{R}^n = \bigcup_{i=1}^{k} \Omega_i^1$ where $c\ell(\Omega_i^1) \subset \operatorname{Int}(\Omega_i)$ and $\Omega_i^1 \cap \Omega_j^1 = \phi$ for all i,j. Define

$$u(x) = u_i(x) \quad \text{if} \quad x \in \Omega_i^1$$

and suppose there is a neighbourhood U of $\bigcup_{i=1}^{k} \partial\Omega_i^1$ so that in U

$$u(x) = \max_{i=1,\ldots,k} u_i(x)$$

then if $u(x,t)$ is the solution of

$$\begin{aligned} u_t &= \Delta u + f(u) \\ u(x,0) &= u(x) \end{aligned} \tag{3.12}$$

it is nondecreasing in t and if $\lim_{t\to\infty} u(x,t) = v(x)$, $v(x)$ is the minimal solution (if such exists) of

$$\begin{aligned} \Delta v + f(v) &= 0 \\ v(x) &\geq u(x) \end{aligned} \tag{3.13}$$

Proof. Let $v(x,t) \equiv u(x)$ then $v(x,t)$ is a subsolution for (3.12). So $u(x,h) \geq u(x,0)$, if $u(x,t)$ satisfies (3.12). Consider the function $w(x,t) = u(x,t+h)$, since it satisfies

$$w_t = \Delta w + f(w) \quad ,$$

by Principle 3.2 $u(x,t+h) \geq u(x,t)$. $u(x,t)$ is therefore nondecreasing and if there is a solution $w(x)$ to (3.13) we obviously must have $u(x,t) \leq w(x)$ and so $\lim_{t\to\infty} u(x,t) = v(x)$ exists and $v(x) \leq w(x)$. It remains only to show that $v(x)$ satisfies $\Delta v + f(v) = 0$.

We have

$$u(x,t+\tau) = K(x,t)*u(x,\tau) + \int_0^t K(x,t-s)*f(u(x,s+\tau))ds \quad .$$

By repeated applications of the dominated convergence theorem

$$v(x) = \lim_{\tau\to\infty} u(x,t+\tau) = K(x,t)*v(x) + \int_0^t K(x,t-s)*f(v(x))ds \quad .$$

Since $v(x)$ is then smooth it must satisfy

$$v_t = \Delta v + f(v) \quad .$$

But v is independent of t, so $v(x)$ satisfies $\Delta v + f(v) = 0$.

This proof is essentially that of Aronson and Weinberger's proposition 2.2 [2].

We now return to the full system of equations (1.1) and define invariant rectangles for the semiflow $S(t)$ on B. Suppose $P = \prod_{i=1}^{m} [a_i,b_i]$ is a rectangle in $\mathbb{R}^m$. P is an invariant rectangle if the set

$$M(P) = \{u(x) = (u_1(x),\ldots,u_m(x)) \mid a_i \le u_i(x) \le b_i, \quad i = 1,\ldots,m$$
$$\text{and all} \quad x \in \mathbb{R}^m\}$$

is positively invariant (see definition 1.4). Weinberger [1] and Chueh, Conley and Smoller [1] have given sufficient conditions for a rectangle to be invariant, we give these as Principle 4.

<u>Principle 3.4</u>. A rectangle $P = \prod_{i=1}^{m} [a_i,b_i]$ is invariant for the semiflow of (1.1) if, for all $i = 1,\ldots,n$

$$f_i(u) \le 0 \quad \text{if} \quad u_i = b_i$$
$$f_i(u) \ge 0 \quad \text{if} \quad u_i = a_i \quad .$$

Remark: The condition says that f does not point out on the boundary of D.

III. GLOBAL EXISTENCE AND SMOOTHING

For equation (1.1) it was remarked in section I that the local time of existence depends only on $\|u\|_\infty$ and so an a priori estimate on $\|u(x,t)\|_\infty$ will suffice to prove global existence. If there is an invariant rectangle P then there is a constant C, so that if $u(x) \in M(P)$ then $\|u(x,t)\|_\infty < C$. This means that $S(t)$ induces a global semiflow on an invariant rectangle. From now, we shall assume that (1.1) admits an appropriate invariant rectangle and the semiflow will be restricted to it.

Now suppose $u(x,t)$ is a smooth solution of (1.1) with $u(x,0) \in M(P)$ on a set $D = \Omega \times [0,T]$ or $\Omega \times [0,\infty)$ where Ω is closed. Let $D^1 \subset \operatorname{int} D$ with D^1 closed, a standard estimate of $\sup\limits_{D^1} |\nabla u|$ in terms of $\sup\limits_{D} |u|$ and information from the equation. As the author does not know of a proof in the literature, short of Schauder estimates which are unnecessary here, we sketch a proof*.

Theorem 3.1. Under these assumptions, there is a constant depending on the distance between D^1 and ∂D and P, say C, so that

$$\sup_{D^1} |\nabla u| \leq C \sup_{D} |u| \tag{3.14}$$

where $|\nabla u| = \sum\limits_i |\nabla u_i|$ etc.

Proof. Let $\zeta(x,t)$ be a smooth cut off function so that

$$\text{if} \quad (x,t) \in D^1 \quad \zeta(x,t) = 1$$

$$\text{if} \quad (x,t) \in D^c \quad \zeta(x,t) = 0$$

and $\zeta(x,t) \in [0,1]$ always. Consider the system of equations

$$v^k_t - \Delta v^k = g^k(x,t) \tag{3.15}$$

on D, where g^k is smooth and let v^k be a smooth solution. Set

$$w = \zeta^2 \sum_k |\nabla v_k|^2 + \lambda \sum_k v_k^2 \quad .$$

Let K be a constant (bigger than 1) that satisfies

$$K > |D^\alpha \zeta|^j$$

---

*I am grateful to Professor L. C. Evans for teaching me this method, it is also related to an estimate in Chueh, Conley and Smoller [1].

where α is a multi index $|\alpha| \leq 2$ and $1 \leq j \leq 2$. At a point of maximum interior to D we must have

$$w_t - \Delta w \geq 0$$

computing and estimating, we get

$$0 \leq \zeta^2(w_t - \Delta w) \leq \zeta^2 \sum_k |\nabla v^k|^2(6K^2 - 2\lambda) + 2\zeta^2\lambda \sum_k v^k g^k$$

$$+ 2\zeta^4 \sum_k \nabla v^k \cdot \nabla g^k + 8K^2\zeta^2 \sum_k |\nabla v^k|^2 + 8\lambda\zeta \sum_k u^k \nabla u^k \cdot \nabla\zeta$$

where the last two terms are arrived at by using the fact that $w_{x_i} = 0$ for all i. For the last term

$$\zeta v^k \nabla v^k \cdot \nabla\zeta = (\zeta\nabla v^k)\cdot(v^k\nabla\zeta) \leq \frac{\zeta^2|\nabla v^k|^2}{2a} + \frac{a(v^k)^2|\nabla\zeta|^2}{2}$$

for any $a > 0$, so let $a = 4$, then

$$0 \leq \zeta^2 \sum_k |\nabla v^k|^2(14K^2 - \lambda) + 2\zeta^2\lambda \sum_k v^k g^k + 2\zeta^4 \sum_k \nabla v^k \cdot \nabla g^k$$

$$+ 16\lambda K \sum_k (v^k)^2 \quad .$$

Now let $g^k(x,t) = f_k(u(x,t))$ and $v^k = u_k$, since $u(x,t) \in M(P)$ for all $t \geq 0$ and f is smooth it can be seen that there are constants L_1 and L_2 so that

$$\sum_k u_k f_k \leq L_1 \sum_k |u_k|^2$$

$$\sum_k \nabla u_k \cdot \nabla f_k(u(x,t)) \leq L_2 \sum_k |\nabla u_k|^2 \quad .$$

Plugging these into the above

$$0 \leq \zeta^2 \sum_k |\nabla u_k|^2((14 + 2L_2)K^2 - \lambda)) + (2K\lambda L_1 + 16\lambda K)\sum_k |u_k|^2 \quad .$$

Taking λ large enough the quantity

$$\frac{2KL_1 + 16\lambda K}{\lambda - (14+2L_2)K^2}$$

is positive and so

$$\zeta^2 \sum_k |\nabla u_k|^2 \leq C \sum_k |u_k|^2 \quad .$$

This is true at a point of maximum, so throughout D

$$\zeta^2 \sum_k |\nabla u_k|^2 + \lambda \sum_k |u_k|^2 \leq (c+\lambda) \sup_D \sum_k |u_k|^2$$

but in D^1 $\zeta = 1$ and (3.14) follows.

As an application we can set $D = \mathbb{R}^n \times [0,\infty)$ and $D^1 = \mathbb{R}^n \times [t_0,\infty)$. If $u(x,t)$ is a smooth solution of (1.1) we get a bound, on $\sup_{\mathbb{R}^n}|\nabla u(x,t)|$ for all $t \geq t_0$ since $u(x,t)$ stays in M.

Suppose we replace the sup-norm topology on B with the compact-open topology, then this says that $\mathcal{M} \cdot [t,\infty)$ is precompact in M for each positive t, where $\mathcal{M}$ consists of smooth functions in M. Since $cl_0(\mathcal{M}) = M$ and

$$cl_0(\mathcal{M}) \cdot [t,\infty) \subset cl_0(\mathcal{M} \cdot [t,\infty)) \subset cl_c(\mathcal{M} \cdot [t,\infty))$$

where cl_0 and cl_c refer to the sup norm and compact open topologies respectively, $M \cdot [t,\infty)$ is precompact in M, with the compact-open topology.

To complete this picture we must show that the topology can be switched without destroying the semiflow property. The only property to be proved is continuity, this will be done in the next section.

IV. CONTINUITY IN THE COMPACT-OPEN TOPOLOGY

Given an invariant rectangle P for the semiflow $S(t)$ on B, when referring to the associated positively invariant set $M(P) \subset B$ we shall assume it is endowed with the compact-open topology.

In this section we show that if P is an invariant rectangle the mapping

$$S : [0,\infty) \times M(P) \to M(P)$$

given by $S(t)u(x) = u(x,t)$ is continuous. We could not replace the topology on all of B and get a continuous mapping. The a priori bound implicit in M is essential.

Suppose $G = \{G_i\}$ is a sequence of compact subsets of $\mathbb{R}^n$ so that $\mathbb{R}^n = \bigcup_{i=1}^{\infty} G_i$ then the following quantity is a metric on M

$$d_G(u,v) = \sum_{i=0}^{\infty} \frac{1}{2^i} \sup_{x \in G_i} |u(x) - v(x)| \quad .$$

To prove continuity of S on $[0,\infty) \times M$ it suffices to prove continuity of $S(t)$ on M which is uniform with respect to t in some

compact interval. This follows from the fact that for fixed u
$S(t)u$ is continuous in t, from section 1.

We will find a sequence of compact sets $G = \{G_i\}$ so that for $0 \le t \le T$ we have an estimate of the form

$$(3.16) \qquad d_G(u(x,t),v(x,t)) \le c(T,N)d_G(u(x),v(x)) + \delta(T,N)$$

with N a positive integer and

$$\delta(T,N) \to 0 \qquad \text{as} \qquad N \to \infty$$

$$c(T,N) \to \infty \qquad \text{as} \qquad N \to \infty \quad .$$

Such an estimate clearly performs the desired function and we prove it in the following theorem.

Theorem 3.2. In the above notation, $S : [0,\infty) \times M \to M$ is continuous.

Proof. Recalling the heat kernel $K(x,t)$ from section I, with an abuse of notation we think of $K(x,t)$ as an $m \times m$ diagonal matrix with this kernel as each diagonal element

$$(3.17) \quad u(x,t) = \int_{\mathbb{R}^n} K(x-y,t)u(y)dy + \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)f(u(y,s))dyds \quad .$$

Let $A \subset B$ both be compact sets and use the notation

$$|g(x)|_A = \sup_{x\in A} |g(x)| = \sup_{x\in A} \Sigma|g_i(x)|$$

$$|u(x,t) - v(x,t)|_A \le \left|\int_B + \int_{B^c} K(x-y,t)[u(y) - v(y)]dy\right|_A$$

$$+ \int_0^t \left|\int_B + \int_{B^c} K(x-y,t-s)(f(u(y,s)) - f(v(y,s)))dy\right|_A ds$$

$$\leq |u(x) - v(x)|_B + c\varepsilon(A,B,t) +$$

$$\int_0^t (k|u(x,s) - v(x,s)|_B + c\varepsilon(A,B,t-s))ds$$

where

$$c > \{\sup |u(x)| : u(x) \in M\}$$

and

$$k > \{\sup |D_{u_i} f(u)| : 1 \leq i \leq m \text{ and } u \in M\}$$

$$\varepsilon(A,B,t) = \int_{B^C} |K(x-y,t)|_A dy \quad .$$

Now let the $\{G_i\}$ be concentric balls of radius R_i. We use the notation $G(N)$ for the sequence of sets $\{G_{i+N}\}_{i=0}^{\infty}$ and $d_{G(N)}$ for the associated metric, then

$$d_{G(N)}(u,v) \leq 2^N d_G(u,v) \quad . \tag{3.18}$$

For each i, substitute $A = G_i$ and $B = G_{i+N}$ and sum over i with the appropriate weighting.

$$d_G(u(x,t),\ v(x,t)) \leq d_{G(N)}(u,v) + c \sum_i \frac{1}{2^i} \varepsilon(G_i,G_{i+N})$$

$$+ \int_0^t (k\ d_{G(N)}(u(x,s),\ v(x,s)) + c \sum_i \frac{1}{2^i} \varepsilon(G_i,G_{i+N}))ds$$

where $\varepsilon(A,B) = \varepsilon(A,B,T)$ and we have used the fact the $\varepsilon(A,B,t)$ is increasing in t. Using (3.18)

$$d_G(u(x,t),\ v(x,t)) \le 2^N d_G(u,v) + c(1+t) \sum_i \frac{1}{2^i} \varepsilon(G_i,G_{i+N})$$

$$+ k\ 2^N \int_0^t d_G(u(x,s),\ v(x,s))ds \quad .$$

By Gronwall's Inequality

$$d_G(u(x,t),\ v(x,t)) \le 2^N e^{k2^N T} d_G(u,v) + c(1+T)e^{k2^N T} \sum_i \frac{1}{2^i} \varepsilon(G_i,G_{i+N}) \quad .$$

In the notation of (3.16) $c(T,N) = 2^N e^{k2^N T}$ and $\delta(T,N) =$ $c(1+T)e^{k2^N T} \sum_i \frac{1}{2^i} \varepsilon(G_i,G_{i+N})$. It remains to show that as $N \to \infty$ $\delta(T,N) \to 0$ for fixed T.

$$\varepsilon(G_i,G_{i+N}) = (4\pi T)^{-n/2} \int_{G_{i+N}^c} \sup_{x\in G_i} \exp(-|x-y|^2/4T)dy$$

recalling that G_i is a ball of radius R_i

$$= (4\pi T)^{-n/2} \int_{G_{i+N}^c} \exp(-(|y|-R_i)^2/4T)dy$$

$$= (4\pi T)^{-n/2} \int_{R_{i+N}}^{\infty} r^{m-1}\exp(-(r-R_i)^2/4T)dr$$

$$= (4\pi T)^{-n/2} \int_{R_{i+N}-R_i}^{\infty} (q+R_i)^{m-1}\exp(-q^2/4T)dq \quad .$$

It is a simple matter to obtain estimates of the form

$$\int_R^{\infty} q^s e^{-cq^2} dq \le p(R)e^{-cR}$$

where $p(R)$ is a polynomial in R of order $s - 1$ (unless $s = 1$, then it is of order 1). These are straightforward if $s = 0,1$, for larger s one integrates by parts and reduces to one of these cases.

From this we see that

$$\varepsilon(G_i,G_{i+N}) \leq R_i^{m-1} p(R_{i+N}-R_i)\exp(-\frac{(R_{i+N}-R_i)}{4T}) \quad .$$

We now choose $R_i = \sum_{j=0}^{i} 3^j$, then

$$R_i = \frac{3^{i+1}-1}{2} \quad , \quad R_{i+N} - R_i = \frac{3^{i+1}(3^N-1)}{2}$$

so

$$\varepsilon(G_i,G_{i+N}) \leq (3^{i+1}-1)^{m-1} p(3^{i+1}(3^N-1))\exp(-\frac{3^{i+1}(3^N-1)}{8T}) \quad .$$

We must show that

$$\lim_{\substack{i\to\infty \\ N\to\infty}} e^{k2^N T}\varepsilon(G_i,G_{i+N}) = 0$$

no matter which order the limits are taken in, since

$$\delta(T,N) = C(1+T)e^{k2^N T} \sum_i 2^{-i} \varepsilon(G_i,G_{i+N})$$

this will finish the theorem. For some constants $c_j(T) > 0$

$$e^{k2^N T}\varepsilon(G_i,G_{i+N}) \leq c_1 3^{(i+1)(m-1)} 3^{(i+N+1)(m-2)} \exp[c_2(-3^{i+1}(3^N-1)) + c_3 2^N]$$

$$= c_1 \exp[\ln 3((i+1)(m-1) + (i+N+1)(m-2)) + c_2(-3^{i+1}(3^N-1)) + c_3 2^N] \quad .$$

If $m \neq 1$, and if $m = 1$ it is even simpler. Dividing the [illegible] in parentheses by 3^{i+N+1} we reach the desired conclusion.

Chapter 4

Spherically Symmetric Solutions

I. MAIN CONTINUATION ARGUMENT

The construction given in section II will allow us to pass structure between different invariant subflows of the semiflow for the reaction - diffusion equation. These subflows will be related to symmetries of the equation, whence their invariance. In particular we shall pass from one space dimensional behaviour to spherically symmetric behaviour.

We firstly express this in the abstract setting of a compact semiflow on a fiber bundle.

Definition 4.1. A semiflow $S(t)$ on a space Y will be called compact if $Y \cdot [t,\infty)$ is precompact in Y for every $t > 0$.

We shall consider a compact semiflow on a fiber bundle but the structure to be continued will lie entirely within a certain subspace. For the purpose of continuation the relevant features of the subspace are given by the following definition.

Definition 4.2. Suppose $\pi : E \to \Lambda$ is a fiber bundle. A subspace $F \subset E$ is said to be continuous if (1) F is closed and (2) $\pi|_F$ is an open map.

Note that any fiber bundle is a continuous subspace of itself. If $E_\lambda = \pi^{-1}(\lambda)$, continuation involves deducing behaviour in the fibers E_λ from that in E_{λ_0}, if λ is close to λ_0. If $A_\lambda = E_\lambda \cap A$, that A is continuous guarantees that this can be done from A_{λ_0} to A_λ.

The pieces of structure we shall continue are isolated invariant sets and attractors (see Conley [1] and Yung [1]).

Let $S(t)$ be a semiflow on Y and N be any closed subset of Y, define the closed set

$$N^0 = \{y \in N : y \cdot [0,\infty) \subset N\} \quad .$$

A set $I \subset Y$ is an isolated invariant set if $I \subset \mathrm{int}(N)$ for some closed set N and $\omega(N^0) = I$, N is then called an isolating neighbourhood for I. An isolated invariant set I is an attractor if it possesses an isolating neighbourhood N with the property that $\omega(N) = I$, such an N is called an attracting neighbourhood.

It is the property of being an isolating (or attracting) neighbourhood that is stable under perturbation not that of being an isolated invariant set or an attractor. If N is an isolating neighbourhood we call the set it isolates $I(N)$. Even though N remains an isolating neighbourhood for nearby semiflows, $I(N)$ may change.

Now suppose we have a compact semiflow $S(t)$ on the total space E of a fiber bundle and that each fiber is invariant. If $F \subset E$ is invariant, that it be a closed subspace of E means the induced semiflow on F is also compact. In the following theorem, N is a closed set with non-empty interior.

<u>Theorem 4.1</u>. Suppose A is a continuous invariant subspace of a bundle, which carries a compact semiflow, if $N_{\lambda_0} = N \cap A_{\lambda_0}$ is an isolating (attracting) neighbourhood in A_{λ_0} then $N_\lambda = N \cap A_\lambda$ is a non-empty isolating (attracting) neighbourhood in A_λ for λ sufficiently close to λ_0.

Proof. If N_{λ_0} is an isolating neighbourhood then there exists a closed set K_{λ_0} and an open set V_{λ_0} both in A_{λ_0} such that

$$\omega(N^0_{\lambda_0}) \subset Int(K_{\lambda_0}) \subset V_{\lambda_0} \subset N_{\lambda_0} \quad .$$

But then there is a closed $K \subset E$ with $int(K) \neq \emptyset$ and an open set $V \subset E$ so that $K_{\lambda_0} = K \cap A_{\lambda_0}$ and $V_{\lambda_0} = V \cap A_{\lambda_0}$, so that $K \subset V \subset N$ and

$$(4.1) \qquad \omega(N^0_{\lambda_0}) \subset Int(K \cap A_{\lambda_0}) \subset V \cap A_{\lambda_0} \subset N_{\lambda_0} \quad .$$

Let $N_A = N \cap A$, since A is continuous $N_A \cap \pi^{-1}(\lambda)$ is non-empty for λ sufficiently close to λ_0. Now $N_A \cap \pi^{-1}(\lambda) = N \cap A_\lambda$ so it suffices to show that

$$(4.2) \qquad \omega(N^0_A \cap \pi^{-1}(B)) \subset Int(K \cap A)$$

for some neighbourhood $B \subset \Lambda$ of λ_0.

Suppose (4.2) were not true then there would be a decreasing sequence of closed sets B_n so that $\cap B_n = \lambda_0$ and (4.2) were false for B_n. For each n, pick a y_n in $\omega(N^0_A \cap \pi^{-1}(B_n)) \cap (N_A \backslash Int(K \cap A))$. Since the semiflow is compact on A, $\omega(N^0_A \cap \pi^{-1}(B_1))$ is compact and since A is closed in E, $\omega(N^0_A \cap \pi^{-1}(B_1)) \cap (N_A \backslash Int(K \cap A))$ is also compact. But then the set $\{y_n\}$ has a limit point y and by the product structure $y \in \pi^{-1}(\lambda_0)$ and clearly also $y \in N_A \cap Int(K \cap A)$ which implies $y \in N_{\lambda_0} \backslash Int(K \cap A_{\lambda_0})$. But $y \in \omega(N^0_A \cap \pi^{-1}(B)) \cap A_{\lambda_0}$. It is not hard to see that $\omega(N^0_A \cap \pi^{-1}(B)) = \bigcup_{\lambda \in B} (N^0_A \cap \pi^{-1}(\lambda))$ and so $y \in \omega(N^0_A \cap \pi^{-1}(\lambda_0)) = \omega(N^0_{\lambda_0})$ but this is a contradiction to (4.1).

The same argument proves the theorem for attracting neighbor-hoods, one only need go through the proof removing all the [illegible] of 0.

A much stronger statement is really true here, namely that the [illegible] generalized Morse index (see Conley [1], Yung [1]) should continue through the fibers of such a subspace but that the above two [illegible] structure continue is sufficient for our purposes.

II. GENERAL APPLICATION

In this section we will construct a product space (i.e. a trivial fiber bundle) which portrays both the spherically symmetric behaviour in (1.1) and its one-dimensional behaviour and further, allows continuation between the two.

We start with the product space $E = M \times [0,1]$ and extend the semiflow on M by leaving the second co-ordinate fixed. We will describe the continuous subspace A, that is of interest, by its fibers.

If the underlying spatial domain in (1.1) is $\mathbb{R}^n$ then fix a co-ordinate system $x = (x_1,\ldots,x_n)$ on it. We describe $A_\lambda = A \cap M \times$ [illegible] in two different cases.

$\underline{\lambda < 1}$ $u \in A_\lambda$ if u is spherically symmetric with the origin considered as being at $(\frac{-\lambda}{1-\lambda}, 0,\ldots,0)$

$\underline{\lambda = 1}$ $u \in A_\lambda$ if u only depends on x_1.

Note that as $\lambda \to 1$, the origin approaches $-\infty$ along the x_1-axis and so the functions in A_λ tend to those in A_1, this is what makes A a continuous subspace and so is the essential content of Proposition 4.1. All the functions in A_λ are functions of essentially one independent variable, if $\lambda < 1$ this is the radial variable, if $\lambda = 1$ it is x_1. For each $\lambda < 1$ let

$$x^\lambda = r^\lambda - \frac{\lambda}{1-\lambda}$$

where r^λ is the radial distance from the origin at $(\frac{-\lambda}{1-\lambda}, 0,\ldots,0)$. If $\lambda = 1$ let $x^\lambda = x_1$. Considered as a function of x, for every λ we have that $x^\lambda = x_1$ on the x_1-axis. Given any $u(x) \in A_\lambda$ it is clear that it can be seen as a function of x^λ so we often write $u(x)$ as $u(x^\lambda)$.

<u>Proposition 4.1</u>. A, as described above, is a continuous invariant subspace of $M \times [0,1] \to [0,1]$.

<u>Proof</u>. A is obviously invariant since Δ is invariant under rigid motions. We need to show that $\pi|_A : A \to [0,1]$ is an open map and that A is closed.

To show that $\pi|_A$ is an open map, an open set C around u has the following lengthy description

$$C = \{(v,\lambda) : |v(x) - u(x)| < \varepsilon \text{ for } x \in K, K \text{ compact}, v \in A_\lambda$$
$$\lambda \in I \text{ where } I \subset [0,1] \text{ is open}\}.$$

If $u \in A_{\lambda_0}$ it must be shown that C intersects all other fibers A_λ for λ sufficiently close to λ_0. This is clearly only a problem if $\lambda_0 = 1$. Suppose $u \in A_1$, then u is a function of x_1 alone, say $u = u(x_1)$. Then consider $v_\lambda \in A_\lambda$ given by $v_\lambda(x^\lambda) = u(x^\lambda)$ if $x^\lambda \geq -\frac{\lambda}{1-\lambda}$. Each v_λ and u agree on the half line $x^\lambda \geq -\frac{\lambda}{1-\lambda}$ so it is clear that for a given compact set K, λ can be chosen sufficiently close to 1 so that $|v_\lambda(x) - u(x)| < \varepsilon$ for $x \in K$.

To show that A is closed, again we need only worry about $u_n(x) \in A_{\lambda_n}$ with $\lambda_n \to 1$. We must show that there is a $u(x) \in A_1$ such that $u_n(x) \to u(x)$ uniformly on compact sets if $u_n(x)$ converges in M. Restricting each $u_n(x)$ to the x_1-axis we get a sequence of functions $u_n^1(x_1)$ that converge to a function $u^1(x_1)$, we define $u(x) = u^1(x_1)$, then it is obvious by the nature of A_{λ_n} that $u_n(x) \to u(x)$ in this topology. This completes the proof.

This seemingly innocent and simple result has fairly strong consequences, for it says that attractors will continue from A_1 to A_λ where $\lambda < 1$. In A_1 we have effectively one-dimensional behaviour, while in A_λ for $\lambda < 1$ we have spherically symmetric behaviour. So if there is an attractor for the one-dimensional equation we automatically obtain a corresponding attractor for the spherically symmetric equation. In other words, some stable behaviour in the one-dimensional case yields associated stable behaviour for the spherically symmetric case.

There is, however, a drawback as we have on A_1 the compact-open topology, which is quite inappropriate for discussing stability. The remedy for this is actually restricting the discussion to another invariant subspace of A in which stable behaviour does occur. As we shall see in the next section, in the application to the bistable equation the topology restricted to the invariant subspace will be close enough to the sup-norm topology to render an attractor.

So, if we have an attractor for the one-dimensional equation the above procedure pulls one in from "infinity" for the spherically symmetric equation but it tells us nothing about the structure of the attractor except that it is the maximal invariant set in some given neighbourhood. For a particular situation the problem then is to describe this maximal invariant set, for it contains the asymptotic information about spherical propagation. In the next section we shall do this analysis for our standing example, the bistable equation.

III. SPECIFIC APPLICATION

We consider again the bistable equation

$$u_t = \Delta u + f(u) \tag{4.3}$$

where $u \in \mathbb{R}$ and f satisfies (H1). We base the set M on the trivial rectangle $P = [0,1]$, which is easily seen to be invariant. Consequently (4.3) generates a global compact semiflow on M which, collecting the pieces, is given by

$$M = \{u : \mathbb{R}^n \to \mathbb{R} \mid u(x) \quad \text{is uniformly continuous and} \quad 0 \le u(x) \le 1\} .$$

The construction of section II of this chapter automatically supplies us with the continuous subspace A of $M \times [0,1]$. As remarked at the end of section II to find an attractor in A_1 we must restrict the space further. In particular we require that a function $u \in A_\lambda$ be nonincreasing in x^λ, which is an invariant condition. Proposition 4.1 was not proved for this A but the proof requires only trivial modifications to cover this case, so A is a continuous subspace of $M \times [0,1]$.

For the one-dimensional version of (4.3)

$$u_t = u_{xx} + f(u) \tag{4.4}$$

we know there is a travelling wave $u(x-ct)$ with $u(-\infty) = 1$, $u(+\infty) = 0$ and $c > 0$, which is monotone decreasing and so lies in A_1. Because of the topology in A_1 this solution appears as a compact curve in A_1 running from 0 to 1,

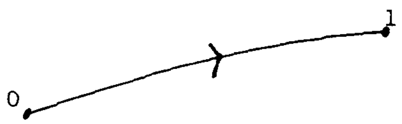

Figure 4.1

let us call this compact set W.

In this section we shall apply a theorem that is proved in chapter 5 which says that W is an attractor in A_1 and analyse its continuation to A_λ.

Theorem 4.3. With respect to the space A_1, $u \equiv 0$ is an isolated invariant set, $u \equiv 1$ and the set W are both attractors. Moreover, for any $\varepsilon < \alpha$ and any $a \in \mathbb{R}$ the set

$$U(a,\varepsilon) = \{u(x_1) \in A_1 \mid u(a) \leq \varepsilon\} \tag{4.5}$$

is an isolating neighbourhood. There exists a $\delta^* < 1$ so that for any $b \in \mathbb{R}$ and $1 > \delta > \delta^*$

$$V(b,\delta) = \{u(x_1) \in A_1 \mid u(b) \geq \delta\} \tag{4.6}$$

is an attracting neighbourhood of $u \equiv 1$. Finally if

$$W \subset U(a,\varepsilon) \cup V(b,\delta)$$

then $U \cup V$ is an attracting neighbourhood of W.

(Notation: as above, the dependencies of U and V will sometimes be suppressed.)

Proof. This follows from Lemmas 5.1, 5.2 and Theorem 5.2 in chapter 5, section 2.

Let us now plug this information into A and analyse the continued sets. Consider the following three subsets of $M \times [0,1]$

$$\mathcal{U}(a,\varepsilon) = \{(u(x),\lambda) \mid u(a,0,\ldots,0) \leq \varepsilon,\ \lambda \in [0,1]\}$$

$$\mathcal{V}(b,\delta) = \{(u(x),\lambda) \mid u(b,0,\ldots,0) \geq \delta,\ \lambda \in [0,1]\}$$

$$N = \mathcal{U} \cup \mathcal{V} \ .$$

Then $\mathcal{U} \cap A_1 = U(a,\varepsilon)$, $\mathcal{V} \cap A_1 = V(b,\delta)$ and $N \cap A_1 = U \cup V$. So if a, ε, b and δ satisfy the hypotheses in Theorem 4.3 we can conclude by Theorem 4.1 that $\mathcal{U} \cap A_\lambda$ is an isolating neighbourhood, $\mathcal{V} \cap A_\lambda$ and $N \cap A_\lambda$ are attracting neighbourhoods for λ sufficiently close to 1. We can clearly write

$$A_\lambda \cap \mathcal{U}(a,\varepsilon) = \{u(x^\lambda) \in A_\lambda \mid u(a) \le \varepsilon\}$$
$$A_\lambda \cap \mathcal{V}(b,\delta) = \{u(x^\lambda) \in A_\lambda \mid u(b) \ge \delta\} .$$

As long as $\lambda < 1$ we are already amongst spherically symmetric functions and so the above statements about $\mathcal{U}$, $\mathcal{V}$ and N can be translated into ones about functions $u : \mathbb{R}^n \to \mathbb{R}$ which depend only on r. We set

$$U_s(a,\varepsilon) = \{u(r) \mid u(a) \le \varepsilon\}$$
$$V_s(b,\delta) = \{u(r) \mid u(b) \ge \delta\}$$

if a and b are large enough U_s is an isolating neighbourhood and V_s is an attracting neighbourhood, as is $U_s \cup V_s$ if $b - a$ is sufficiently large. We assume that these conditions on a and b are satisfied and analyse the invariant sets.

<u>Proposition 4.2</u>. $I(V_s(b,\delta)) = 1$ if δ is close enough to 1 and b is sufficiently large.

<u>Proof</u>. One can check easily that under these circumstances one of the comparison functions in Aronson and Weinberger [2], see our theorem 1.1, will be less than every element of $V_s(b,\delta)$.

Since the semiflow is compact and $U_s \cup V_s$ is connected, $\omega(U_s \cup V_s)$ is connected and since $0 \in U_s$, $I(U_s) \neq \phi$. The usual argument then shows that $\omega(U_s \cup V_s)$ contains $I(U_s)$, 1 and orbits running from the former to the latter. In particular $I(U_s)$ is not an attractor, however we do have the following proposition; recall that A_0 is just the set of spherically symmetric functions which are non-increasing in r.

Proposition 4.3. $u \equiv 0$ is an attractor in A_0.

Proof. It is a standard maximum principle argument that $u \equiv 0$ is an attractor in the sup-norm topology. Take any compact set $K \subset \mathbb{R}^n$ which contains the origin, if functions in A_0 satisfy an estimate on K they satisfy it everywhere and so a neighbourhood in the compact open topology is contained in a sup-norm neighbourhood which 0 attracts.

This proposition shows that $I(U_s)$ contains something other than just $u \equiv 0$. However we can show that the flow on $I(U_s)$ induced by the semiflow of the equation is a gradient flow, that is there is a functional $V(u)$ which is decreasing along orbits except on constant solutions. The functional $V(u)$ is the standard energy functional

$$V(u) = \int_{\mathbb{R}^n} (\tfrac{1}{2}\,|\nabla u|^2 - \int_0^u f(s)ds)dx \quad . \tag{4.7}$$

It is a standard computation to show that along orbits

$$\frac{d}{dt}\,V(u(x,t)) = -\int_{\mathbb{R}^n} u_t^2 dx \quad . \tag{4.8}$$

The only reason (4.7) is not used more often is that the integrals are often unbounded. However, here we can show that $I(U_s) \subset H^1(\mathbb{R}^n)$

which is in the domain of $V(u)$ and so it inherits a ...

<u>Theorem 4.4</u>. $I(U_s) \subset H^1(\mathbb{R}^n)$.

<u>Proof</u>. We actually show much more, namely if we define the set

$$U^T = \{u \in U_s \mid u \cdot t \in U_s \text{ for all } t \le T\}$$

then $U^T \cdot T$ is contained in some set Z^T which is closed and consists of functions that are exponentially decreasing. Since $I(U_s) \subset U^T \cdot T$ for all $T \ge 0$, elements of $I(U_s)$ are exponentially decreasing and we can then estimate the derivatives to prove the proposition.

We will define a supersolution to the equation $\bar{u}(r,t)$ with the following properties

$$\bar{u}(r,0) \ge \varepsilon \tag{4.9}$$

$$\lim_{t\to\infty} \bar{u}(r,t) \le C\, e^{-mr} \quad \text{some constants } C,\ m > 0 \tag{4.10}$$

$$\bar{u}(a,t) \ge \varepsilon \quad \text{for all } t \ge 0 \ . \tag{4.11}$$

Suppose that these three properties can be satisfied, then the function $\min\{\alpha, \bar{u}(r,t)\}$ is also a supersolution on the unbounded annulus $r \ge a$. If $u \in U^T$ then $u(a,t) \le \varepsilon$ for $t \le T$ and since $u(r,0) = u(r) \le \varepsilon$ for all r

$$u(r,T) \le \min\{\alpha, \bar{u}(r,T)\} \ .$$

So define Z^T as the set of $v(r) \in A_0$ that satisfy

$$v(r) \le \min\{\alpha, \bar{u}(r,T)\} \ .$$

From (4.10) $\bigcap_{T\geq 0} Z^T$ consists of exponentially decreasing functions.

To define $\bar{u}(r,t)$ we use a method of Fife and McLeod's [1] from their Lemma 4.1. Let $v(x)$ be any solution of $v_{xx} + f(v)$ that is nonconstant but $\lim_{x\to\pm\infty} v(x) = 0$, i.e. the body of the fish in Figure 3, that also satisfies $v(a) > \varepsilon$. Set

$$\bar{u}(r,t) = v(r-p(t)) + ce^{kt}$$

where c, k and $p(t)$ are to be chosen. If $p(t) \geq 0$ for $t \geq 0$ and $p(t)$ is bounded as $t \to \infty$, $c \geq \varepsilon$ and $k < 0$ then 4.9 - 11 are satisfied.

We only need show that $\bar{u}(r,t)$ is a supersolution if $\bar{u} \leq \alpha$ and only on the half-line $[c,\infty)$ where $v(c) = \alpha$ and $v'(c) < 0$. We compute

$$(4.12)\qquad \begin{aligned} L\bar{u} &= \bar{u}_t - \bar{u}_{rr} - \frac{1}{r}\,\bar{u}_r - f(\bar{u}) \geq \bar{u}_t - \bar{u}_{rr} - f(\bar{u}) \\ &= -p'v' + kce^{kt} - v'' - f(v + ce^{kt}) \quad . \end{aligned}$$

Consider two sets separately $u \in [0,\delta]$ and $u \in (\delta,\alpha]$ where, for constants q_1 and q_2

$$f'(u) < q_1 < 0 \quad \text{for} \quad u \in [0,\delta]$$

$$v'(r) < q_2 < 0 \quad \text{for} \quad u \in (\delta,\alpha] \quad .$$

From (4.12)

$$L\bar{u} \geq -p'v' + kce^{kt} - (f(v + ce^{kt}) - f(v)) \quad .$$

If $\bar{u} \in [0,\delta]$

$$L\bar{u} \geq -p'v' + e^{kt}(kc - q_1 c) \quad .$$

If $p'(t) \geq 0$ then

$$L\bar{u} \geq ce^{kt}(k - q_1)$$

so pick $k < 0$ but $k > q_1$ then $L\bar{u} \geq 0$. If $\bar{u} \in [\delta,\alpha]$ there is a K so that

$$L\bar{u} \geq -p'q_2 + ce^{kt}(k - K)$$

so we can set

$$p' = \frac{c(k-k)}{q_2} e^{kt}$$

since $k < K$ and $q_2 < 0$, $p' \geq 0$, we can then set

$$p(t) = \frac{c(k-K)}{kq_2} e^{kt} + z$$

to get $L\bar{u} \geq 0$ for all $\bar{u} \in [0,\alpha]$. Pick z so that $p(t) \geq 0$ for $t \geq 0$, clearly $p(t) \to z$ as $t \to \infty$ and choose $c = \varepsilon$.

From this it follows that $I(U_s)$ consists of exponentially decreasing functions and therefore these functions are in $L^2(\mathbb{R}^n)$. To estimate $|\nabla u(x)|$ for $u(x) \in I(U_s)$, take $\Omega = \{r \mid r \geq R_0\}$ and $\Omega' = \{r \mid r \geq R_1\}$ with $R_0 > R_1$ then from Theorem 3.1 with $D = \Omega \times [0,\infty)$ and $D' = \Omega' \times [0,\infty)$

$$\sup_{D'} |\nabla u| \leq c \sup_{D} |u|$$

and so

$$\sup_{\Omega} |\nabla u| \leq c \sup_{\Omega'} |u|$$

for any $u \in I(U_s)$. Since c depends only on $R_0 - R_1$ and $|u|$ is exponentially decreasing it follows that $|\nabla u|$ is also, which proves the theorem.

That $I(U_s)$ inherits the semiflow as a gradient flow means that it consists of equilibrium solutions and orbits connecting them. All the equilibrium solutions must be ones that would satisfy the boundary conditions in Theorem 3.1.

If we suppose that f satisfies (H2), then there is a unique non-constant equilibrium solution satisfying $u(+\infty) = 0$ (Theorem 3.3). In this case $I(U_s)$ consists of $u \equiv 0$, this equilibrium solution, call it $\bar{u}$, and orbits connecting them.

In $\omega(U_s \cup V_s)$ there are orbits running from $I(U_s)$ to 1. Let $u(r,t)$ be such an orbit, then $\lim_{t\to-\infty} u(r,t) \subset I(U_s)$. If t is a large enough negative number we can show that $u(r,t)$ is exponentially decreasing in r and $u(r,t) \in H^1(\mathbb{R}^n)$. Pick T, a large enough negative number so that $u(r,t) \in U_s$ for all $t \leq T$, then there is a sequence u_n so that $u_n \cdot t_n \to u(r,T)$ and $u_n \cdot t \in U_s$ for all $t \leq t_n$ so $u_n \cdot t_n \in Z^{t_n}$ but then $u(r,T) \in \bigcap_{T \geq 0} Z^T$ and the above follows as before.

A consequence of the fact that $u(r,t) \in H^1(\mathbb{R}^n)$ if t is large negative is that $\lim_{t\to-\infty} u(r,t)$ is an equilibrium point in $I(U_s)$ and now this limit can be considered in the L^2 norm or the sup-norm.

Suppose again that f satisfies (H2) then the only point [illegible] solution $u(r,t)$ can approach in backward time is $\bar{u}$. Moreover, by Theorem 2.3 we know the spectrum of the linearized problem around $\bar{u}$ and so we can apply the invariant manifold theorems in infinite dimensions. For the equation

$$u_t = \Delta u + f(u) \tag{4.13}$$

the linearized operator around $\bar{u}$ acting on v is

$$\Delta v + f'(\bar{u})v = Lv \quad . \tag{4.14}$$

We consider $\sigma(L)$ with respect to the space $L^2(\mathbb{R}^+)$ with the measure $r^{n-1}dr$, then it is standard that $\sigma(L) \subset \mathbb{R}$ and the essential spectrum is contained in $(-\infty,b]$ where $b = f'(0)$ and so $b < 0$ (by H1). But then from Theorem 2.3, $\sigma(L) \cap [0,\infty)$ consists of only one point λ and $\lambda > 0$. Then from Theorem 5.2.1 in Henry [1], there is locally at $\bar{u}$ a one-dimensional unstable manifold which consists of all solutions that approach $\bar{u}$ in negative time.

From this we see that there are only two solutions leaving $\bar{u}$, one must go to 0 and the other to 1. So if f satisfies (H2) we have a complete description of the spherical attractor W_s, pictorially.

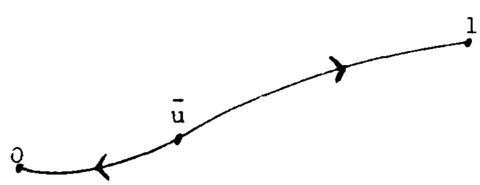

Figure 4.2

Chapter 5

Spherically Symmetric Solutions in a Moving Frame

I. EXTENSION OF CONTINUATION ARGUMENT

The space $A \subset M \times [0,1]$ of chapter 4, section II inherited its semiflow from $M \times [0.1]$ which comes from the equation in question. In this chapter we endow A with a family of semiflows that do not leave the fibers invariant. Studying the behaviour in A_1 and its relationship to the rest of these semiflows will give much finer information about the propagating solutions. We will not present this application until section III of this chapter, but the construction of the semiflow should serve as motivation for the generality of the theorems and propositions in section II. The statements in this section are to be understood in the context of a general system of reaction - diffusion equations, but the remainder of the chapter is exclusively about the bistable equation (1.3) where f satisfies (H1).

Let us call the basic semiflow on M, $S(t)$. We shall define a semiflow, called $H(t)$, on the space $A \times [0,\infty)$. Firstly we define a translation map on A,

$$T : \mathbb{R} \times A \to A$$

by the formula (recall $A \subset M \times [0,1]$)

$$T(a,\ u(x_1,\ldots,x_n),\lambda) = (u(x_1+a,\ x_2,\ldots,x_n),\lambda') \tag{5.1}$$

where

$$\lambda' = \frac{a + (\lambda/_{1-\lambda})}{1 + a + (\lambda/_{1-\lambda})} \ . \tag{5.2}$$

It is clear that if $u(x_1,\ldots,x_n) \in A_\lambda$ then $u(x_1+a, x_2,\ldots,x_n) \in A_{\lambda'}$, where λ' is given by (5.2), so T is well defined. Since translation is continuous on M and λ' is a continuous function of λ and a, from inspection of (5.2), $T : \mathbb{R} \times A \to A$ is continuous. Intuitively T translates functions in the x_1 direction and moves them over to the appropriate fiber. Notice, from (5.2) that A_1 is invariant under T, as it should be.

For $c \in [0,\infty)$, we use T to define a semiflow $H_c(t)$ on A. Let $(u(x),\lambda) \in A$, define

(5.3) $$H_c(t)(u(x),\lambda) = T(ct)(S(t)u(x),\lambda) \quad .$$

If $c > 0$, as $t \to \infty$, $H_c(t)$ pushes all of A onto A_1, intuitively it pushes the origin out to $-\infty$. Consequently the asymptotic analysis for this semiflow will depend on its behaviour in A_1. $H_c(t)$ gives the evolution of the equation but viewed while moving out in a radial direction with speed c. Restricted to A_1, the semiflow is that of the one-dimensional equation in a moving co-ordinate frame, so we will see how the asymptotic behaviour of the spherical solutions is determined by the one-dimensional equation at different speeds.

For each $t \geq 0$, define $H(t)$ on $A \times [0,\infty)$ by

$$H(t)((u,\lambda),c) = (H_c(t)(u,\lambda),c) \quad .$$

It is clear that H defines a semiflow on $A \times [0,\infty)$ that contains $H_c(t)$ by restricting to $A \times \{c\}$.

We must turn now to analysing the asymptotic behaviour of the one-dimensional bistable equation in various moving co-ordinate frames.

II. ATTRACTOR IN ONE DIMENSIONAL CASE

For the standing example we assume again that f satisfies (H1) (see chapter 1). In A_1 there are functions of a single variable, call it ξ, which are nonincreasing in ξ and stay between 0 and 1. The semiflow $H_c(t)$ when restricted to A_1 is that of the equation

$$u_t = u_{\xi\xi} + cu_\xi + f(u) \tag{5.4}$$

where $\xi = x_1 - ct$, so this agrees with previous notation when $c = 0$.

The set W is the union of 0, 1 and all translates of the travelling wave, which is invariant under each of the semiflows $H_c(t)$. The main goal of this section is to prove that W is an attractor for $c \in [0, c^*]$, some $c^* > \bar{c}$, relative to A_1.

The strongest stability results for the travelling wave are due to Fife and McLeod [1]. Specialising some of their results to the present case reads as follows.

Theorem 5.1. (Fife and McLeod). If f satisfies (H1) and $0 \le u(x) \le 1$ is uniformly continuous, $x \in \mathbb{R}$, and satisfies

$$\liminf_{x\to-\infty} u(x) > \alpha \quad \text{and} \quad \limsup_{x\to+\infty} u(x) < \alpha$$

then there is a constant x_0 such that

$$\lim_{t\to\infty} \| u(x,t) - u_1(x - \bar{c}t + x_0) \|_\infty = 0$$

where $u_1(\xi)$ is the one-dimensional travelling wave. Further, the limit is uniform over a sup-norm neighbourhood of u_1.

This could be described as uniform asymptotic stability of the travelling wave. Fife and McLeod actually prove a much stronger result, namely exponential stability, but we shall not use this.

Before proving that W is an attractor, the local dynamical properties of the constant solutions $u \equiv 0$ and $u \equiv 1$ must be studied.

In the topology of A_1 a closed neighbourhood of 0, i.e. one with non-empty interior containing 0, is a set which depends on a compact set K and a positive function $g(\xi)$ defined on K. Such a set is then defined as (recalling $A_1 \subset M$)

$$U(K,g) = \{u \in A_1 \mid u(\xi) \leq g(\xi) \quad \text{for} \quad \xi \in K\} \quad .$$

A neighbourhood of 1 is given similarly

$$V(K,g) = \{v \in A_1 \mid v(\xi) \geq g(\xi) \quad \text{for} \quad \xi \in K\} \quad .$$

If $K = \{a\}$, then we write $U(a,\varepsilon)$ and $V(a,\varepsilon)$ where $\varepsilon = g(a)$, this is consistent with the notation of chapter 4.

Throughout this chapter we shall apply the comparison principles of chapter 3, section II to the equation (5.4), it is trivial to extend each one to cover this case.

Lemma 5.1. If $\varepsilon < \alpha$, for any a, in the c-semiflow on A_1, $u \equiv 0$ is an

(A) isolated invariant set if $\bar{c} > c > 0$

(B) attractor if $c > \bar{c}$

and $U(a,\varepsilon)$ is an isolating (in case (A)) and attracting (in case (B)) neighbourhood.

Proof. To deal with Case (A) define

$$U_c^0(a,\varepsilon) = \{u \in U(a,\varepsilon) : u\cdot_c[0,\infty) \subset U(a,\varepsilon)\}$$

where "$\cdot_c$" refers to the action of the c-semiflow. From chapter 3, it suffices to show that $\omega_c(U_c^0(a,\varepsilon)) = \{0\}$, where ω_c refers to the ω-limit set in the c-semiflow. This will follow if we can show that given any neighbourhood U of 0 there is a T so that $U_c^0(a,\varepsilon)\cdot_c t \subset U$ for all $t \geq T$. By Comparison Principle II it is obviously adequate to find a solution to (5.4) so that if $v(\xi) \in U_c^0(a,\varepsilon)$ then

$$v(\xi) \leq u(\xi,0)$$

and $u(\xi,t) \to 0$ in A_1.

From theorem 5.1, if $c < \bar{c}$, then $v(\xi) \in U_c^0(a,\varepsilon)$ implies that $\lim_{\xi\to-\infty} v(\xi) \leq \alpha$. So we need only construct a solution with $u(\xi,0) \geq \alpha$ if $\xi < a$ and $u(\xi,0) \geq \varepsilon$ if $\xi \geq a$.

Transforming (5.4) into a system yields

$$\text{(5.5)} \qquad \begin{aligned} w' &= z \\ z' &= -cz - f(w) \end{aligned} \quad .$$

If $c = 0$, the phase portrait is in Figure 2.1, for $c > 0$ it is

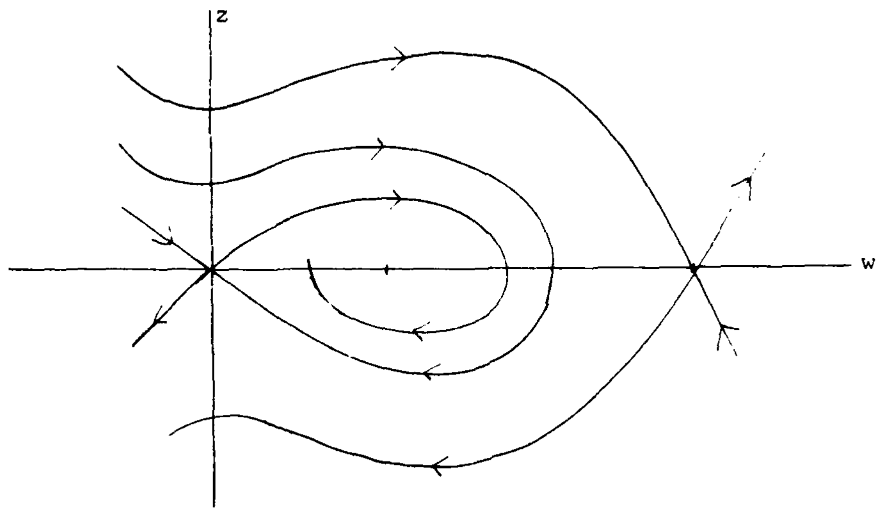

Figure 5.1

The important fact here is that for any $0 < \varepsilon < \alpha$ the solution with $w(0) = \varepsilon$ and $z(0) = 0$ must cross the line $w = \alpha$ in the lower half plane, let d be the point that gives $w(d) = \alpha$ and $z(c) < 0$. Now define $u(\xi)$ by

$$u(\xi) = \begin{cases} \alpha & \text{if } \xi \le a + d \\ w(\xi - a) & \text{if } a + d \le \xi \le a \\ \varepsilon & \text{if } \xi \ge a \end{cases} .$$

Then $u(\xi)$ satisfies the hypotheses of Comparison Principle III and so clearly $u(\xi,t) \to 0$ as $t \to \infty$ uniformly on compact sets. But also if $\xi \le a$, $u(\xi,0) = u(\xi) = \alpha$ and $u(\xi,0) \ge \varepsilon$ if $\xi \ge a$. So $u(\xi,t)$ performs the desired function.

Now suppose $c > \bar{c}$, for case (B), to show 0 is an attractor we need $\omega_c(U) = \{0\}$. As before we construct $u(\xi)$ so that $v(\xi) \in U$ implies $v(\xi) \le u(\xi)$, and $u(\xi,t) \to 0$ in A_1. But by theorem 5.1

any function in A_1 which satisfies (i) $\varepsilon < \lim_{x \to +\infty} u(\xi) < \alpha$ and (ii) $u(\xi) = 1$ if $\xi \leq a$ (of which there are obviously many) will have $u(\xi,t) \to 0$ in A_1, since such a function will obviously majorise U, the Lemma is proved.

<u>Lemma 5.2</u>. There is a $c^* > \bar{c}$ so that $u \equiv 1$, in the c-semiflow on A_1, is an

(A) attractor if $0 < c < \bar{c}$

(B) isolated invariant set if $c^* > c > \bar{c}$

and if $1 > \delta > \alpha$ then, for any b, $V(b,\delta)$ is an isolating neighbourhood in case (A). For each c in (B), there is a δ so that $V(b,\delta)$ is an attracting neighbourhood, for any b.

<u>Proof</u>. The proof is very similar to that of Lemma 5.1. In case (A) a function $u(\xi)$ below everything in $V(b,\delta)$ can easily be found if $\delta > \alpha$ and by Theorem 5.1 $u(\xi,t) \to 1$ in A_1.

For case (B), by Theorem 5.1, V_c^0 consists of functions $v(\xi)$ which satisfy $v(\xi) \geq \alpha$ for all ξ. If c^* is not too much bigger than $\bar{c}$ the unstable manifold of $(1,0)$ intersects the line $w = \alpha$ in the lower half-plane for the system (5.5) with $\bar{c} < c < c^*$.

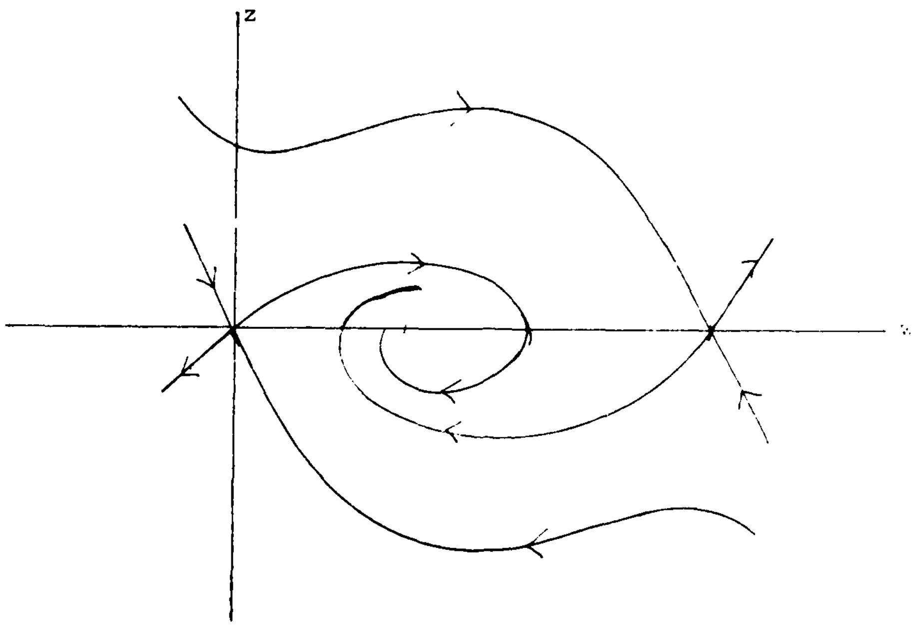

Figure 5.2

So for each c, $\bar{c} < c < c^*$ there is a $\delta = \delta(c)$ such that if $(w(\xi), z(\xi))$ satisfies $w(0) = \delta$, $z(0) = 0$ then there exists $d > 0$ so that $w(d) = \alpha$ and $z(d) < 0$. Now define $u(\xi)$ by

$$u(\xi) = \begin{cases} \delta & \text{if} \quad \xi \leq a - d \\ w(\xi - (a-d)) & a \geq \xi \geq a - d \\ \alpha & \text{if} \quad \xi \geq \alpha \ . \end{cases}$$

$u(\xi)$ satisfies the hypotheses of Comparison Principle III and so $u(\xi,t) \to 1$ in A_1. But also if $v(\xi) \in V_c^0(b,\delta)$ then $v(\xi) \leq u(\xi)$. This completes the proof.

Remark: The conditions under which Lemma 5.2 gives an isolating neighbourhood of $u \equiv 1$ are a little cumbersome, but the following three simple points will help.

(1) If $V(b,\delta)$ is an isolating neighbourhood for a given $c > \bar{c}$ then it is for all c' which satisfy $c < c' < \bar{c}$.

(2) If $V(b,\delta)$ is an isolating neighbourhood for $c > \bar{c}$, then so is $V(b,\delta')$ if δ' satisfies $1 > \delta' > \delta$.

(3) If the solution to (5.5) with $z(0) = 0$, $w(0) = \delta$ crosses the line $u = \alpha$ in the lower half-plane then $V(b,\delta)$ is an isolating neighbourhood for the c-semiflow.

Before proving that W is an attractor we must consider some slightly more exotic neighbourhoods of 0 and 1. For each $c \in [0,c^*]$, where c^* is the same as in Lemma 5.2, and each $\varepsilon < \alpha$ sufficiently close to α the solution of (5.5) with $w(0) = \varepsilon$ and $z(0) = 0$ must cross the w-axis between α and 1 in backward time. Let $d < 0$ be the largest number such that $z(d) = 0$ and set $\delta = w(d)$. Let $K_a = [a,a-d]$ and $g_a(\xi) = w(\xi - (a-d))$ with domain K. Suppose $e-d < a$, then we will use the sets $U_c(K_a,g_a)$ and $V_c(K_e,g_e)$ given by

$$U_c(K_a,g_a) = \{u \in A_1 \mid u(\xi) < w(\xi - (a-d)) \quad \text{for} \quad a \leq \xi \leq a-d\}$$

$$V_c(K_e,g_e) = \{u \in A_1 \mid u(\xi) > w(\xi - (e-d)) \quad \text{for} \quad e \leq \xi \leq e-d\} \quad .$$

The picture is

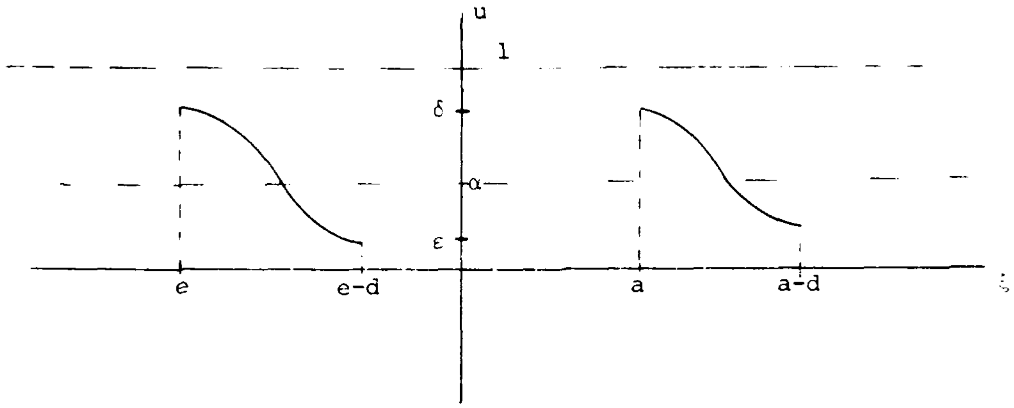

Figure 5.3

$U_c(K_a,g_a)$ consists of functions below the right hand curve and $V_c(K_e,g_e)$ of those above the left hand curve.

For each $c \in [0,c^*]$ these sets are isolating neighbourhoods of 0 and 1 respectively as they are contained in sets of the form found in Lemmas 5.1 and 5.2. Namely, $U_c(K_a,g_a) \subset U(a-d,\varepsilon)$ and $V_c(K_e,g_e) \subset V(e,\delta)$, the fact that w exists implies that δ is close enough to 1 for $V(e,\delta)$ to be an isolating neighbourhood by remark (3) after Lemma 5.2. The crucial property of these sets is that they give us a positively invariant neighbourhood of W.

Lemma 5.3. For any $0 < \varepsilon < \alpha$ and any $c \in [0,c^*]$, if $e-d < a$ then the set

$$U_c(K_a,g_a) \cup V_c(K_e,g_e)$$

is positively invariant in the c-semiflow.

Proof. Let $u(\xi) \in U_c(K_a,g_a)$ then we must show that $u(\xi,t) \in U_c(K_a,g_a) \cup V_c(K_e,g_e)$ for all positive t. So there are two things to prove:

(1) If $u(\xi,\tau) \in U_c(K_a,g_a)$ but $u(\xi,t) \notin U_c(K_a,g_a)$ for $t > \tau$ but close to τ then $u(\xi,\tau) \in V_c(K_e,g_e)$ and (2) the same as (1) with the roles of U_c and V_c reversed. The proofs of (1) and (2) are, not surprisingly, almost identical, so we only prove (1).

Now suppose $u(\xi,t)$ is a solution of (5.3) with $u(\xi,\tau) \in U_c(K_a,g_a)$. Since g_a on K_a is a solution of (5.3) itself then $u(\xi,t) \in U_c(K_a,g_a)$ for all $t \le \tau$ unless, by comparison principle I, $u(\xi,t) \in U_c(K_a,g_a)$ for $\tau \le t \le \tau_0$ and $u(\xi,\tau_0) = g_a(\xi)$ for $\xi \in \partial K_a$, i.e. $\xi = a$ or $\xi = a-d$. Since $u \in A_1$ is nonincreasing, if $u(a,\tau_0) = g_a(a) = \delta$ we have $u(\xi,\tau_0) \in V_c(K_e,g_e)$.

Suppose $u(a-d,\tau_0) = w(0) = \varepsilon$ then the following inequality holds

$$u_{\xi\xi}(a-d,\tau_0) + cu_\xi(a-d,\tau_0) + f(u(a-d,\tau_0)) \le w_{\xi\xi}(0) + cw_\xi(0) + f(w(0))$$

the last terms on each side are equal. Since $w_\xi(0) = 0$ and $u_\xi \le 0$, if the inequality were not true we would have

$$u_{\xi\xi}(a-d,\tau_0) > w_{\xi\xi}(0)$$

but this is impossible since $u(a-d,\tau_0) = w(0)$, $u_\xi \le 0$ and $u(\xi,\tau_0) \le w(\xi)$ for $\xi < 0$ and close to it.

The above inequality implies that $u_t(a-d,\tau_0) \le 0$ and so this does not provide an escape from $U_c(K_a,g_a)$. This completes the proof of Lemma 5.3.

We will show that the set $U_c \cup V_c$ of Lemma 5.3 is an attracting neighbourhood for W for each c-semiflow, the travelling wave (with 0 and 1 attached). But for the sake of hygiene, we give a more easily described set that is contained in $U_c \cup V_c$. In the definition of U_c and V_c, $\delta = w(d)$, so δ depends on c and setting $\delta = \delta(c)$, define

$$\bar{\delta} = \sup\{\delta(c) \; : \; c \in [0,c^*]\} \quad .$$

which is clearly less than 1. With $b = e-d$, if $\delta \geq \bar{\delta}$

$$U(a,\varepsilon) \cup V(b,\delta) \subset U_c(K_a,g_a) \cup V_c(K_e,g_e)$$

for all c which satisfy $0 < c < c^*$. Note also that if b is sufficiently smaller than a then $W \subset U(a,\varepsilon) \cup V(b,\delta)$. We are free to choose $\varepsilon < \alpha$ as this is free to be chosen in U_c and V_c, but δ must satisfy $\delta \geq \bar{\delta}$, which depends on ε. We will see in chapter 6 that it is useful to have no restriction on ε, the restriction on δ is unimportant.

<u>Theorem 5.2</u>. If $c \in [0,c^*]$ then W is an attractor in the c-semiflow and there exists a $\delta > \alpha$ so that for any $\varepsilon < \alpha$ if

$$W \subset U(a,\varepsilon) \cup V(b,\delta)$$

then $U \cup V$ is an attracting neighbourhood for W in each of these semiflows.

<u>Proof</u>. Consider $\omega_c(U \cup V) = D_c$ since $U \cup V \subset U_c \cup V_c$ (the dependencies on K_a, a etc. are being suppressed, it is assumed they satisfy the requirements in the preamble for the theorem) we have $D_c \subset \omega_c(U_c \cup V_c) \subset U_c \cup V_c$, by Lemma 5.3. $0 \in D_c$ and $1 \in D_c$.

Suppose a solution $u(\xi,t)$ is in D_c, then it is defined for all $t \in \mathbb{R}$ and $u(\xi,t) \in U_c \cup V_c$ for all t. (Recall that $\xi = x_1 - ct$.)

Consider the case $c < \bar{c}$, either (A) $u(\xi,t) \to 0$ as $t \to \infty$ or (B) $u(\xi,t) \to 1$ as $t \to \infty$. If (A) happens we cannot have $\lim_{t\to-\infty} u(\xi,t) = 0$ unless there exists a τ such that $u(\xi,\tau) \notin U_c$, i.e. $u(\xi,\tau) \in V_c$ but then $\lim_{t\to\infty} u(\xi,t) = 1$ since V_c is an attracting neighbourhood of 1, which contradicts (A). So (A) implies that $\lim_{t\to-\infty} u(\xi,t) = 1$, which is impossible, so (A) is impossible and (B) holds. We cannot have $\lim_{t\to-\infty} u(\xi,t) = 1$ as it is an attractor, so $\lim_{t\to-\infty} u(\xi,t) = 0$.

By analogous arguments, if $c > \bar{c}$, we must have $\lim_{t\to+\infty} u(\xi,t) = 0$ and $\lim_{t\to-\infty} u(\xi,t) = 1$.

Since the same statements obviously hold for $\omega_c(U_c \cup V_c)$, a consequence is that for $c \neq \bar{c}$

$$\omega_c(U \cup V) = \omega_c(U_c \cup V_c) \quad .$$

The inclusion from left to right is trivial but if $u(\xi,t) \in \omega_c(U_c \cup V_c)$ for all t then $\lim_{t\to-\infty} u(\xi,t) = 0$ (if $c < \bar{c}$) and since U is a neighbourhood of 0, $u(\xi,t)$ must be in $\omega_c(U \cup V)$ as well, it is a similar argument for $c > \bar{c}$.

Suppose $v(\xi) \in D_{c_1}$ then clearly $v(\xi+a)$ will also be in D_{c_1} $\lim_{t\to-\infty} v(\xi,t) = \lim_{t\to-\infty} v(\xi+a,t)$. Also there is a $v_T(\xi) \in D_{c_1} \subset U_{c_1} \cup V_{c_1}$ such that

$$V(\xi) = H_{c_1}(T)V_T(\xi)$$

but then $V_T\ (\xi - (c_2-c_1)T) \in D_{c_1}$ and

$$H_{c_2}(T)V_T(\xi - (c_2-c_1)T) = T((c_2-c_1)T)H_{c_1}(T)V_T(\xi - (c_2-c_1)T)$$

$$= H_{c_1}(T)T((c_2-c_1)T)V_T(\xi - (c_2-c_1)T)$$

$$= H_{c_1}(T)V_T(\xi)$$

so

$$V(\xi) = H_{c_2}(T)V_T(\xi - (c_2-c_1)T)$$

this implies $v(\xi) \in \omega_{c_2}(U_{c_1} \cup V_{c_1})$, but by the same argument as above $\omega_{c_2}(U_{c_1} \cup V_{c_1}) = \omega_{c_2}(U_{c_2} \cup V_{c_2}) = \omega_{c_2}(U \cup V)$. So $v(\xi) \in D_{c_1}$ implies $v(\xi) \in D_{c_2}$ if $c_i \neq \bar{c}$ $i = 1,2$, call this common set D. We must show that $D_{\bar{c}} = D$. Let $u(\xi,t)$ be an orbit in $D_{\bar{c}}$, then either $u(\xi,t_n) \in U_{\bar{c}} \cap V_{\bar{c}}$ for a sequence $t_n \to -\infty$, $u(\xi,t) \in U$ *for all large negative* t or $u(\xi,t) \in V$ for all large negative t. In the former case, it is a simple consequence of Theorem 5.1 that $u(\xi,t)$ would be the travelling wave and so in D. In either of the latter cases $H_c(t)u(\xi,0)$ would lie in U or V respectively for some $c \neq \bar{c}$ and large negative t. This is either impossible or it puts $u(\xi,t)$ in D.

The proof will be complete if we can show that $D = W$. We consider D in the $\bar{c}$-semiflow (for the sake of notation, we shall drop the $\bar{c}$ in "$\cdot_{\bar{c}}$").

Define the set $s(U)$, for a given set U, to be the set of $u(\xi)$ where $u(\xi+s) \in U$. We firstly show that if $u \in D$ either (1) $u \in s(U_c \cap V_c)$, for some s, or (2) $u(\xi) \leq \varepsilon$ for all ξ or (3) $u(\xi) \geq \delta$ for all ξ.

Suppose neither (2) nor (3) are true then there is a ξ so that $\varepsilon < u(\xi) < \delta$. It must be true that $\lim_{\xi\to+\infty} u(\xi) < \varepsilon$ and $\lim_{\xi\to-\infty} u(\xi) > \delta$, otherwise some translate of $u(\xi)$ would not lie in $U_c \cup V_c$ and this is impossible as D is translation invariant (argued above for D_c, $c \neq \bar{c}$) and $D \subset U_{\bar{c}} \cup V_{\bar{c}}$.

It follows that the set

$$G = \{p \in \mathbb{R} \mid u(\xi) \in s(V_{\bar{c}})\}$$

is non-empty. Let $s = \inf G$, then $s \in G$ as V_c is closed. We claim that $u(\xi) \in s(U_{\bar{c}})$, if this were not true there would exist a p so that both $u(\xi) \notin p(V_{\bar{c}})$ and $u(\xi) \notin p(U_{\bar{c}})$ by minimality of s and the fact that $U_{\bar{c}}$ is closed. But this is a contradiction as we would then have $u(\xi+p) \notin U_{\bar{c}} \cup V_{\bar{c}}$, and $u(\xi+p) \in D \subset U_{\bar{c}} \cup V_{\bar{c}}$.

So one of the alternatives (1), (2) or (3) above is true of $u(\xi) \in D$. If either (2) or (3) were true, since D is invariant, we would have $u(\xi) \in \{0,1\}$. So we have shown that

(5.6) $$D \subset \{0,1\} \cup \Big(\bigcup_{s\in\mathbb{R}} s(U_{\bar{c}} \cap V_{\bar{c}})\Big)$$

But it follows easily from theorem 5.1 that $\omega_{\bar{c}}\big(\bigcup_{s\in\mathbb{R}} s(U_{\bar{c}} \cap V_{\bar{c}})\big) = W$ and so $\omega_{\bar{c}}(D) = W$. Since $\omega_{\bar{c}}(D) = D$, $D = W$.

From above $D = \omega_c(U \cup V)$ for any $c \in [0,c^*]$ and this [illegible] the proof of the theorem.

Remark: (1) In lemmas 5.2 and 5.3, 0 and 1 are not [illegible] for the case $c = \bar{c}$, this is because they obviously are not [illegible] in that semiflow, the travelling wave is a string of critical [illegible] that approaches both of them. (2) From Lemmas 5.1, 5.2 and Theorem [illegible] we have all we need for the theorems of chapter 4, section III [illegible] specialising to the case $c = 0$. Note however that we did not [illegible] this case in isolation, the above proof depended crucially on using the spectrum of semiflows.

III. APPLICATION OF EXTENDED CONTINUATION

Again we consider the bistable equation (1.3) where f satisfies (H1) ((H2) is not needed in this section at all).

We will apply the semiflow of section I and the results of section II to the question of how solutions behave when followed out in a radial direction with speed c. If $u(r,t)$ is a solution, for a given c, we want to study its behaviour along lines $r - ct = \text{constant}$. So for any $p \in \mathbb{R}$ set $r = p+ct$ and we get a function $v(p,t) = u(p+ct,t)$ whose domain is time dependent i.e. $p \geq -ct$. However, for each $p \in \mathbb{R}$ there is a T so that if $t \geq T$ $v(p,t)$ is defined and so it makes sense to try and determine

$$(5.7) \qquad \lim_{t\to\infty} u(p+ct,t)$$

for every $p \in \mathbb{R}$ and this represents moving out in a radial direction with speed c. We can use the semiflow constructed in section I of this chapter to determine (5.7).

From section I we have a semiflow $H(t)$ on the space $A \times [0,c^*]$. Let us consider the results of section II in this context.

In $A_1 \times [0,c^*]$ there is a two-dimensional invariant manifold, namely $W \times [0,c^*]$ with the following flow on it

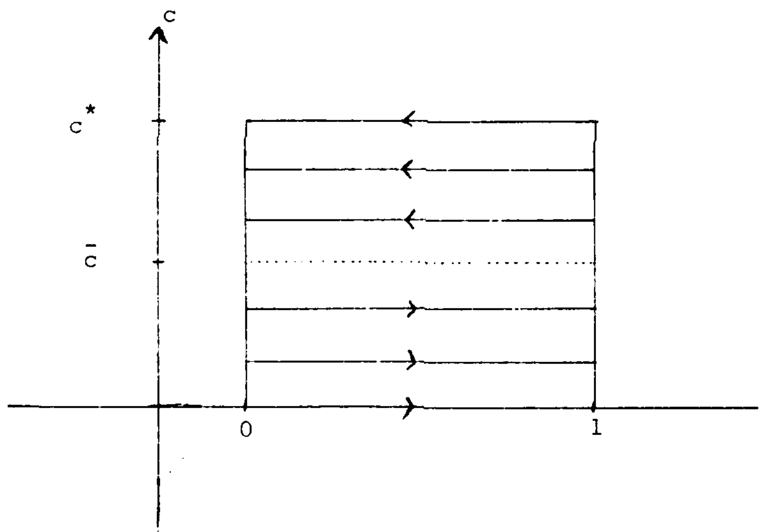

Figure 5.4

where $\bar{c}$ is the speed of the one-dimensional wave, as usual. This manifold is an attractor in $A_1 \times [0,c^*]$, by theorem 5.2, with an attracting neighbourhood given by

$$[U(a,\varepsilon) \cup V(b,\delta)] \times [0,c^*]$$

where a, ε, b, δ satisfy all the conditions given in section II.

We will compute (5.7) for initial data $u(r) \in A_0$ that satisfy

$$\lim_{r\to+\infty} u(r) < \alpha \tag{5.8}$$

(5.9) $u(r,t) \to 1$ as $t \to \infty$, uniformly on compact sets.

<u>Theorem 5.3</u>. Suppose $u(r) \in A_0$ satisfies (5.8) and (5.9) then there is a function $\varphi(t) = o(t)$ so that

$$\lim_{t\to\infty} \|u(p+\bar{c}t+\varphi(t),t) - u_1(p)\|_\infty = 0 \tag{5.10}$$

where u_1 is the one-dimensional travelling wave and its speed is $\bar{c}$.

Remarks: (1) The theorem says that any spherically symmetric solution (with the restriction of being in A_0) which propagates (condition (5.6)) takes the shape of the one-dimensional travelling wave as $t \to +\infty$.

(2) The function $u(p+\bar{c}t+\varphi(t),t)$ is actually defined on a time-dependent domain of p's. For the norm to make strict sense we can extend the function to be constant for $p \leq -\bar{c}t-\varphi(t)$.

(3) An alternative way to stating (5.10) would be

$$\lim_{t\to\infty} \|u(r,t) - u_1(r-\bar{c}t-\varphi(t))\|_\infty = 0 \quad , \tag{5.11}$$

where the supremum is now taken over $r \in \mathbb{R}^+$. This is how the statement of theorem 5.1 was made. It is clear that (5.10) and (5.11) are equivalent.

The main idea in the proof of theorem 5.3 is a shooting argument for the semiflow $H(t)$. From knowing the behaviour of $\lim_{t\to+\infty} H_c(t)u$ for $c < \bar{c}$ and $c > \bar{c}$ we shall make a conclusion about $\lim_{t\to+\infty} H_{\bar{c}}(t)u$. (Notation: we shall denote an element of A by the component that is a function in M, the λ-co-ordinate is implicit in the symmetry of this function.)

We will need a perturbation statement similar to theorem 4.1 but for this extended semiflow on $A \times [0,\infty)$. A neighbourhood of a function $v(x) \in A_1$, depending on a compact set K, contained in the x_1-axis, and an $\varepsilon > 0$, is of the form

$$\{u(x) \mid |u(x)-v(x)| \leq \varepsilon \text{ for } x \in K\} \quad . \tag{5.12}$$

Let U be such a set, or a union of such sets. Let $\bar{U} \subset M$ be defined by the same inequalities. Set

$$\mathcal{U} = (\bar{U} \times [0,1]) \cap A \tag{5.13}$$

$$U^\lambda = \mathcal{U} \cap (M \times [\lambda,1]) \quad . \tag{5.14}$$

(It is to be understood that if a Roman letter refers to a set, the corresponding script letter refers to the associated set given by (5.13). Similarly, if a superscript λ appears.)

Let $I \subset [0,\infty)$ be compact and $N = U \times I \subset A \times [0,\infty)$. N and N^λ have their obvious meanings. In the following "$\cdot$" refers to the action of $H(t)$ on $A \times I$.

<u>Lemma 5.4</u>. If, for some $\tau > 0$

$$c\ell(N \cdot [\tau,\infty)) \subset \mathrm{Int}(N) \tag{5.15}$$

then there is a $\lambda < 1$ so that

(5.16) $$c\ell(N^{\lambda} \cdot [\tau,\infty)) \subset \text{Int}(N^{\lambda}) \quad .$$

Proof of Lemma 5.4. It suffices to show that for each $\tau_2 \geq \tau_1 > 0$, there is a $\lambda < 1$ so that

(5.17) $$c\ell(N^{\lambda} \cdot [\tau_1,\tau_2]) \subset \text{Int}(N^{\lambda}) \quad .$$

Setting $\tau_1 = \tau$, $\tau_2 = 2\tau$ in (5.17) and iterating yields (5.16).

We can pick an open set V and a closed set C, in A_1, so that $V \subset C \subset \text{Int}(N)$ and

(5.18) $$c\ell(N \cdot [\tau_1,\tau_2]) \subset V \quad .$$

Now suppose there is no λ for which (5.17) is satisfied. Then there is a sequence $\{u_n\}$ with $u_n \in U^{\lambda_n}$, $\lambda_n \to 1$, sequences $\{c_n\}$ and $\{t_n\}$ so that $(u_n,c_n) \cdot t_n \notin V$.

$H(t)$ is a compact semiflow as compactness was proved by a global derivative estimate which is obviously preserved under translation. It follows that $\{(u_n,c_n) \cdot t_n\}$ has a limit point. We can assume that $(u_n,c_n) \cdot t_n \to u$, $c_n \to c'$ and $t_n \to t'$. Using the fact that $\{(u_n,c_n) \cdot \tau_1\}$ is precompact, it is a standard argument to see that $(u_n,c_n) \cdot t_n$ and $(u_n,c') \cdot t'$ have the same limit.

Let $\{v_n\}$ be a sequence in U so that $u_n - v_n \to 0$, for instance let v_n agree with u_n on the x_1-axis above $-\lambda_n/1-\lambda_n$. In theorem 3.2 we actually proved that the semiflow is uniformly continuous with respect to initial data, this is a consequence of estimate (3.16).

It follows that this is also true for $H(t)$ on A. It is then easy to check that $(v_n,c') \cdot t' \to u$, but $u \notin V$ and so this contradicts (5.18).

Proof of Theorem 5.3. Let $u(r) \in A_0$ satisfy (5.8) and (5.9). Recalling c^* from section II, if c_* is between 0 and $\bar{c}$, set $I = [c_*,c^*]$ and $E = u \times I$.

There is a set $N = (U \cup V) \times I$ which is an attracting neighbourhood for the two dimensional invariant manifold of Figure 5.4. By Lemma 5.4, there is a $\lambda < 1$ and $\tau > 0$ so that $c\ell(N^\lambda \cdot [\tau,\infty)) \subset \mathrm{Int}(N^\lambda)$. Since $c^* > 0$, $\omega(N^\lambda) \subset A_1$ and so N^λ is an attracting neighbourhood.

If b is large enough $T(b)E \subset N^\lambda$ by condition (5.8). Consequently $\omega(T(b)E) \subset W \times [c_*,c^*]$.

Let d be a metric on M. Pick any β so that $0 < \beta < d(0,1)$. The sphere $S = \{u \mid d(u,1) = \beta\}$ separates M into two open sets, its interior and exterior $D = (S \times [0,1]) \times [c_*,c^*]$ also separates $A \times [c_*,c^*]$ into two disjoint open sets, with 0 in one of them and 1 in the other. If we can show that for large t $H(t)(T(b)E)$ intersects both these sets, then by connectedness it must intersect D.

We will show that for any fixed $b \geq 0$

$$(5.19) \qquad \lim_{t\to+\infty} H_c(t)(T(b)u) = \begin{cases} 1 & \text{if } c < \bar{c} \\ 0 & \text{if } c > \bar{c}\,. \end{cases}$$

If (5.19) is true for one particular b it is true for every b, by translation invariance. To prove the second part, with the c-semiflow on A, 0 is an attractor by lemma 5.1 with U as an attracting neighbourhood. By lemma 5.4 there is a U^{λ} which is an attracting neighbourhood for 0. If b is large enough $T(b)u \in U^{\lambda}$ and so the limiting behaviour follows.

For the first part, consider the set $G = \{u(r,t) \mid u(r,0) = u(r)\}$. By lemmas 5.2 and 5.4 there is an attracting neighbourhood of 1, V^{λ}, for the c-semiflow if $0 < c < \bar{c}$. If b is large enough $T(b)G \cap V^{\lambda} \neq \emptyset$, since $1 \in c\ell(T(b)G)$ by (5.9) and translation invariance. But then $\lim_{t\to+\infty} H_c(t)(T(b)g) = 1$ for some $g \in G$. Since $g = u(r,t)$, for some T, the first part of (5.19) must hold.

We now know that for each large t, $H(t)(T(b)G)$ intersects D. Then there is a function $c(t)$ such that $H_{c(t)}(T(b)u) \in D$. As $t\to+\infty$, $H_{c(t)}(T(b)u) \to W \times [c_*, c^*]$, but D is closed and there is only one point in W that is a distance β from 1, so $H_{c(t)}(T(b)u)$ converges to this point, which is not 0 or 1 and therefore is a travelling wave, call it u_1.

Rewriting this quantity by restricting to the x_1-axis and letting $\gamma(t) = c(t)t$, we get

$$\lim_{t\to+\infty} u(x_1 + \gamma(t), t) = u_1(x_1) \quad . \tag{5.20}$$

(5.20) is actually in the compact-open topology but since everything is nonincreasing in x_1 (if it is $\geq -\gamma(t)$) then (5.20) also holds in the sup-norm. Letting $\gamma(t) = \bar{c}t + \varphi(t)$ and $p = x_1$ we have (5.10). That $\varphi(t) = o(t)$ is an easy consequence of (5.19).

Remarks: (1) What is most interesting in Theorem 5.3 is that only knowledge of one-dimensional behaviour is used. We prove nothing directly about the spherically symmetric solutions except by perturbation from their one-dimensional limiting behaviour.

(2) It is not hard to see that $\varphi(t) \to -\infty$ as $t \to +\infty$, so the spherical wave may lag behind the one-dimensional wave, but does not get ahead. This follows from the fact that any solution to the one-dimensional equation is a supersolution for the spherically symmetric equation and such a solution can easily be found which majorises $u(r)$ and tends to the travelling wave (by Theorem 5.1).

Chapter 6

Conclusion

I. INTERPRETATION OF RESULTS

In the first two sections of this chapter, we shall discuss exclusively the bistable equation (1.3) with the nonlinearity satisfying (H1) and (H2).

At the end of chapter 4 we found a set, called W_s, that is an attractor in the semiflow on A_0 (the world of spherically symmetric data). It is perhaps not clear what the fact that this set is an attractor means for the behaviour of the solutions of the equation. In this section we shall give an interpretation of this statement in more traditional terms.

Recall that A_0 consists of functions in B that are spherically symmetric, between 0 and 1, nonincreasing in r. Define the set

$$U = \{u(r) \in A_0 \mid u(+\infty) < \alpha\} \quad .$$

W_s being an attractor translates into the following theorem.

Theorem 6.1. $U = U_0 \cup U_{\bar{u}} \cup U_1$ where

$$U_0 = \{u(r) \mid u(r,t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{uniformly}\}$$

$$U_{\bar{u}} = \{u(r) \mid u(r,t) \to \bar{u} \quad \text{as} \quad t \to \infty \quad \text{uniformly}\}$$

$$U_1 = \{u(r) \mid u(r,t) \to 1 \quad \text{as} \quad t \to \infty \quad \text{uniformly on compact sets}\} \quad .$$

Further U_0 and U_1 are both open and connected.

Proof. Given any $u(r)$ U, there is an a and an ε so that $u \in U_s(a,\varepsilon)$, pick a corresponding $V_s(b,\delta)$ then $U_s \cup V_s$ is an attracting neighbourhood for W_s

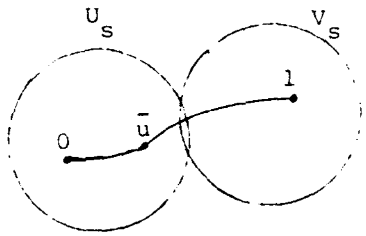

Figure 6.1

and so everything in $U_s \cup V_s$ tends to one of 0, $\bar{u}$ and 1, therefore so does $u(r)$, it is obvious that in the first two cases the convergence is uniform.

Since 0 and 1 are attractors in their own right, U_0 and U_1 must both be open and since neighbourhoods of 0 and 1 are connected they must also be connected. This completes the proof.

Since U is obviously connected the theorem says that the removal of $U_{\bar{u}}$ disconnects the set into these two components that give the propagation and decay behaviour respectively. So $U_{\bar{u}}$ separates the two regimes and, in some sense, gives the optimal set of comparison functions for both propagation and decay. This is, practically speaking, not particularly useful as these are hard to locate except near $\bar{u}$ but they do give an exact picture of the situation.

Another way to express these facts is that $U_{\bar{u}}$ is a codimension one set and so it is "small". The two pieces it separates are the regimes of propagation and decay respectively.

II. ASYMPTOTIC STATES AND PERMANENT SOLUTIONS

We have not quite shown that the propagation regime U_1 is an SAS with respect to A_0 (see definition 1.4). It is open in the compact-open topology and so also in the sup-norm but it would require an estimate on how $\varphi(t)$ varied across initial data to show that elements of U_1 had the same sup-norm asymptotic behaviour, (5.10) is not quite enough. If a definition were formulated which bore the relation to definition 1.2 that orbital stability does to stability in qualitative ordinary differential equations, U_1 would be an SAS in this other sense.

A_0 has three, albeit invariant, conditions on it. As far as U_1 being an asymptotic state is concerned, it is not hard to drop the condition $0 \leq u(r) \leq 1$, and it may be possible to drop the monotonicity condition. But the major open question is whether the condition of spherical symmetry is necessary. For instance it is not known whether there could be a wave that propagates in $\mathbb{R}^2$ with an elliptical wave-front.

For the above, the only feature of W_s used is that it contains three equilibrium points and orbits joining them. We did not use the fact that there are only two connecting orbits in it. This was proved in chapter 4. In terms of theorem 6.1, there is one in U_0 and one in

U_1. Each of these connecting orbits is a permanent solution (see definition 1.3). Consequently there is a unique permanent solution in each SAS U_0 and U_1.

These permanent solutions are "special" solutions of the equation as we would generally only expect forward existence in parabolic equations. Nevertheless, it is not clear what the real significance of their being unique is. Such permanent solutions as equilibria and travelling waves supply the asymptotic information for their SAS. If we consider the SAS U_1 this information is given by the one-dimensional travelling waves, not this permanent solution in U_1. This solution may however be relevant to the asymptotic behaviour of non-spherically symmetric data since it contains spherical information and the travelling waves do not.

III. COMMENTS ON THE TECHNIQUE

The main technique in this work is the construction of the space A which relates the one-dimensional and spherically symmetric behaviour of a reaction - diffusion equation. The space realises the intuition of the one-dimensional world living out "at infinity" in the spherical world. By constructing a concrete mathematical object, such as A, we can see exactly what is forced on the spherically-symmetric solutions by this relationship.

The presentation of this construction here treats this as a homotopy-continuation method. But, in many respects, it is more closely related to the technique in 'Ordinary Differential Equations' of pasting a manifold onto phase space which represents some limiting behaviour. From studying the induced flow on this manifold, conclusions can be made, by continuity, about the flow on the original phase space. A recent example of the power of this technique is in M[illegible] [1], and this work is partially responsible for inspiring the use of this idea here.

It is a drawback in the technique that the underlying topology on A is the compact-open topology because stability statements are more easily made in the sup-norm topology. This topology is used in two essential ways, for compactness of the semiflow and to make the space A tie together so that perturbation is possible from A_1 to $A_{[illegible]}$.

The way to circumvent the problem is to restrict the semiflow to an invariant subspace. This subspace should have the property that an open neighbourhood of the solution whose stability is in question with the inherited topology should be effectively the same as that set with the sup-norm topology. For example, if the stable one-dimensional behaviour expected is given by a travelling pulse or front then the invariant subspace should consist of functions that look like the wave as $x \to \pm\infty$. So the invariant subspace makes the data be close to the wave at $\pm\infty$ and consequently being close in the compact-open topology implies the same in the sup-norm topology. This also shows the necessity of considering an interval of different speeds. The wave can move, metaphorically, "out of sight" in the compact-open topology unless we focus on it at the correct speed.

So the key to applying the technique lies in finding exactly the right places where the space should be tightened to the sup-norm topology or loosened to the compact-open topology.

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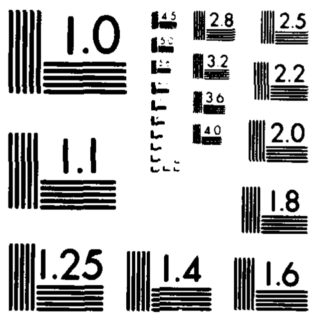

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Attention is focused on the scalar reaction-diffusion equation: $u_t = \Delta_x + f(u)$, where f is cubic-like and $f(0) = f(1) = 0$. Amongst spherically symmetric solutions it is proved there is a bounded unstable equilibrium which is decreasing in radial profile. Under a concavity assumption on f this equilibrium is unique. Moreover there is a unique expanding spherical wave which is defined for all time, positive and negative. As $t \to -\infty$ this wave approaches the unstable equilibrium.

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ABSTRACT (continued)

Aronson and Weinberger [2] have proved before that, in all space dimensions, there are non-trivial solutions that propagate ($u(x,t) \to 1$ uniformly on compact sets as $t \to +\infty$) and ones that decay ($u(x,t) \to 0$ uniformly as $t \to +\infty$). This suggested the existence of the unstable equilibrium.

There is an interesting global description of this propagation/decay effect. The set of initial data whose associated solutions approach the unstable equilibrium as $t \to +\infty$ splits a natural set of functions into two sets. Data from one set yields a solution that propagates, and data from the other set, a solution that decays. This fact is closely related to the uniqueness of the expanding spherical wave.